LEAVING CERTIFICATE EXAMINATION, 1981

MATHEMATICS - HIGHER LEVEL - PAPER II (300 marks)

MONDAY, 15 JUNE - MORNING 9.30 to 12.00

Attempt **QUESTION 1** (100 marks) and **FOUR** other questions (50 marks each)

Marks may be lost if all your work is not clearly shown

1. (i) If $k, n \in \mathbf{N}$, find the maximum value of k for which

$$\sqrt{14^2 + 112^2} = k\sqrt{n}\ .$$

(ii) Find the least number of years in which a sum of money will more than double itself at 10% per annum, compound interest.

(iii) Write $1{\cdot}4\dot{7} = 1{\cdot}4777\ ...$ in the form $\frac{a}{b}$ where $a, b \in \mathbf{N}$.

(iv) The nth term of a series is given by

$$T_n = (1 - n)^2.$$

Find the sum of the first 20 terms.

(v) Differentiate from first principles the function $x \rightarrow \sin x$.

(vi) Differentiate $\dfrac{x}{1 + x^2}$ with respect to x.

(vii) A sequence is defined by

$$x_1 = \tfrac{1}{2}$$
$$(1 + x_r)\ x_{r+1} = (1 - x_r)\ ,\ r = 1, 2, \ldots$$

Find the sum of the first 24 terms.

(viii) A function $y = f(x)$, for $x \in \mathbf{R}$ is defined as follows:

$$y = \sqrt{4 - x^2}\ , \qquad 0 \leqslant x \leqslant 2$$
$$y = x - 2\ , \qquad x \geqslant 2\ .$$

Indicate the graph if the function for $x \geqslant 0$.

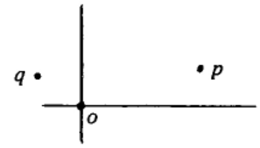
(ix) A sequence is said to be monotonic increasing if $T_n + 1 > T_n$ for all n.
The nth term of a sequence is $T_n = 2 - \frac{1}{n}$
Test if the sequence is monotonic increasing.

(x) $\bar{x}$ and σ are the mean and the standard deviation of the set x_1, x_2, x_3.
Find the mean and the standard deviation of the set

$$\frac{x_1 - \bar{x}}{\sigma}\ ,\quad \frac{x_2 - \bar{x}}{\sigma}\ ,\quad \frac{x_3 - \bar{x}}{\sigma}\ .$$

OR

(x) $\vec{p}$ and $\vec{q}$ are two vectors as in diagram. Express $\vec{qp}$ in terms of $\vec{p}$ and $\vec{q}$ and hence use the scalar product $\vec{qp} \cdot \vec{qp}$ to prove the cosine rule for the triangle opq.



2. (a) $\dfrac{1 + 2i}{1 - i}$ in a root of the quadratic equation

$$ax^2 + bx + 5 = 0 ,$$

where $a, b \in \mathbf{R}$ and $i = \sqrt{-1}$. Find the values of a and b.

(b) If $1, z_1, z_2, z_3, z_4$ are the roots of

$$z^5 - 1 = 0 ,$$

show that $z_1 + z_2 + z_3 + z_4 = -1$ and deduce that

$$\sum_{n=1}^{4} \left(\cos \frac{2n\pi}{5} + i \sin \frac{2n\pi}{5} \right) = -1 .$$

(c) If $z = x + iy$ and $\bar{z} = x - iy$, prove that the image of $x + iy$ under the transformation

$$z \rightarrow \tfrac{1}{2}(z + \bar{z}) - \frac{i}{2}(z - \bar{z})$$

is a real number.

Indicate on the Argand diagram the set

$$|z| \leqslant 1$$

and find the length of its image under the above transformation.

3. (a) Find $\lim\limits_{x \rightarrow 0} \dfrac{(2 - x)^3 - 8}{x}$.

(b) Prove by induction, or otherwise, that

$$1^2 + 2^2 + \ldots + n^2 = \frac{n(n + 1)(2n + 1)}{6} .$$

(c) a^2, x, b^2 are in arithmetic sequence.
a^2, y, b^2 are in geometric sequence where $y > 0$.
Express x and y in terms of a and b and hence prove that

$$x \geqslant y .$$

Deduce that

$$(a + b)\left(\frac{1}{a} + \frac{1}{b}\right) \geqslant 4 .$$

4. (a) Find the value of the derivative of each of the following at the given value of x:

(i) $\sqrt{\dfrac{\sin x}{1 - \cos x}}$ at $x = \dfrac{\pi}{2}$

(ii) $\log (1 + 2x^2)^3$ at $x = \frac{1}{2}$

(iii) $e^{1 - \sin x}$ at $x = \pi$.

(b) If $x = \sin t$ and $y = \sin nt$, express $\dfrac{dy}{dx}$ in terms of t and hence prove that

$$(1 - x^2) \left[\frac{dy}{dx}\right]^2 = n^2 (1 - y^2)$$

for $-1 < x < 1$.

5. The height h cm of a right circular cone is increasing at the rate of $\frac{1}{\pi}$ cm per second. The slant height remains constant and equal to 9 cm.
Find in terms of h the rate at which the volume of the cone is increasing and calculate this rate when $h = 4$ cm.

Deduce the value of h for which the volume is a maximum and calculate this maximum in terms of π.

6. (a) (i) Using division, or otherwise, find $\int_1^2 \frac{x^3 + x^2 + x + 1}{x^2 + 1}\, dx$.

(ii) Evaluate $\int_0^{\frac{\pi}{2}} \cos x \cos 2x \, dx$

(iii) Evaluate $\int_{-1}^{1} \frac{x}{x^2 + 1}\, dx$.

(b) Find the volume generated by rotating the graph of

$$\frac{x^2}{2^2} + \frac{y^2}{1^2} = 1$$

about the x-axis.

7. (a) $\frac{(n + 1)(n + 2)}{(n + 3).(n + 4)}$ is the n-th term of a sequence.

Test the sequence for convergence.

$\frac{(n + 1)(n + 2)}{(n + 3)(n + 4)}$ is the n-th term of a series.

Test the series for convergence.

(b) Test for convergence the series

$$\sum_{n=1}^{\infty} \frac{(x + 2)^n}{n \,.\, 4^n} \quad \text{for } x > 0.$$

(c) $u_1 + u_2 + u_3 + \ldots + u_n + \ldots$ is a convergent series of positive terms such that

$$\lim_{n \to \infty} \frac{u_n}{u_{n+1}} = k > 1.$$

Illustrate on a diagram that for $e > 0$

$$1 < k - e < \frac{u_n}{u_{n+1}} \quad \text{(for large } n\text{)}$$

and hence deduce that

$$u_{n+1} + u_{n+2} < u_n(\alpha + \alpha^2)$$

where $\alpha < 1$.

8. (a) Prove that $P(E \cup F) = P(E) + P(F) - P(E \cap F)$ where $P(X)$ is the probability of the event X.

If E and F are independent events where $P(E) = \frac{1}{2}$ and $P(F) = \frac{3}{4}$, find the value of $P(E \cup F)$.

(b) Two children A and B visit a sweet shop every day between 1 p.m. and 1.30 p.m. and each stays for 6 minutes. A arrives x minutes after 1 p.m. and B arrives y minutes after 1 p.m.

If $x < y$, show that $y < x + 6$ is the condition that the children meet.

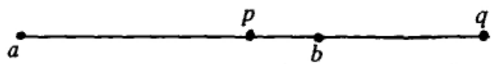
If $y < x$, find the condition that the children meet.

Using the same axes and scales plot these four inequalities for $0 < x < 30$ and $0 < y < 30$ and hence find the probability that the children meet.

Over a period of 100 days find the probability that the children meet on at least 42 occasions.

OR

8. (a) If r is any point in the line ab, prove that $\vec{r} = t\vec{b} + (1 - t)\,\vec{a}, \quad t \in \mathbf{R}$



Given that $|ap| : |pb| = m : n$, express $\vec{p}$ in terms of $\vec{a}$ and $\vec{b}$.
Given that $|aq| : |qb| = m : n$, express $\vec{q}$ in terms of $\vec{a}$ and $\vec{b}$.
If the origin o is taken on the perpendicular bisector of $[ab]$, prove that

$$\vec{op} \cdot \vec{oq} = |\vec{oa}|^2$$

where $\vec{op} \cdot \vec{oq}$ means the scalar product of $\vec{op}$ and $\vec{oq}$.

(b) The points $u, v, w,$ represent the vectors $\vec{i} - 2\vec{j}, \quad -3\vec{i} + \vec{j}$ and $3\vec{i} + k\vec{j},$ respectively. If the three points are collinear, find the value of k.

(c) In any Δabc prove that $\vec{a} + \vec{b} + \vec{c}$ is the orthocentre of the triangle when the circumcentre of the triangle is taken as origin.

$4\vec{i} + 5\vec{j}$ is the circumcentre of a Δabc where $\vec{a} = \vec{i} + 9\vec{j}, \quad \vec{b} = 7\vec{i} + \vec{j},$ $\vec{c} = 4\vec{i} + 10\vec{j}$.

Find in terms of $\vec{i}$ and $\vec{j}$ the orthocentre of the triangle and draw a rough diagram to illustrate your answer.