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LEAVING CERTIFICATE EXAMINATION, 1980

MATHEMATICS - HIGHER LEVEL - PAPER II (300 marks)

MONDAY, 16 JUNE - MORNING 9.30 to 12.00

Attempt **QUESTION 1** (100 marks) and **FOUR** other questions (50 marks each)

1. (i) Simplify $(\sqrt{5} + \sqrt{3})^2 - (\sqrt{5} - \sqrt{3})^2$.

(ii) If $A(1{\cdot}1)^{-1} + A(1{\cdot}1)^{-2} + A(1{\cdot}1)^{-3} = 1000$, find A correct to the nearest integer.

(iii) The nth term of a series is given by $T_n = n(n+1)$. Using the formulae for Σn^2 and Σn, find the sum of the first 29 terms of the series.

(iv) Differentiate $\dfrac{1}{x+1}$ with respect to x from first principles.

(v) If $y = x \sin x^2$, find $\dfrac{dy}{dx}$.

(vi) Write down the first 5 terms of the sequence defined by

$$x_1 = 1, \qquad x_{n+1} = 2x_n, \qquad n \in \mathrm{N}_0.$$

(vii) Find the volume of the solid formed by rotating the graph of $y = x(3-x)$ for $0 \leqslant x \leqslant 3$ about the x-axis.

(viii) The series $\displaystyle\sum_{n=1}^{\infty} \frac{n}{k^n}$ converges when the constant $k > 1$. Use this fact to find

$$\lim_{n \to \infty} \frac{n\,2^n}{3^n}$$

(ix) Say why $y = 1$ is an asymptote of the graph of $y = \dfrac{x+1}{x-1}$ and write down the equation of the other asymptote. By considering the values of y when $x > 1$, $x < 1$, $x < -1$, draw a rough graph (no calculus necessary).

(x) Verify that the two vectors $5\vec{i} + 12\vec{j}$ and $12\vec{i} - 5\vec{j}$ are perpendicular and then write the vector $8\vec{i} + 3\vec{j}$ as a sum of two vectors, one of which is parallel to $5\vec{i} + 12\vec{j}$ and the other perpendicular to it.

OR

(x) If m and σ are the mean and standard deviation of x_1, x_2, x_3, show that $m - 1$ is the mean of $x_1 - 1, x_2 - 1, x_3 - 1$ and express the standard deviation in terms of σ .

2. (a) If ki is a root of $3z^3 - z^2 + 12z - 4 = 0$, find the values of $k \in \mathbf{R}$ and the other root. $(i = \sqrt{-1})$

(b) State De Moivre's Theorem and use it

(i) to express $(1 + i)^{100}$ as a real number,

(ii) to prove $(\sin\theta + i\cos\theta)^9 = \sin 9\theta + i\cos 9\theta$.

(c) If z is a complex number, illustrate on the Argand diagram the set A of z which satisfies

$$|z - 5 - 4i| = 4.$$

Name the transformation

$$z \to \tfrac{1}{2}(z - \bar{z})$$

and hence, or otherwise, illustrate the image of the set A under this transformation.

3. (a) Let $k = \lim\limits_{n \to \infty} \dfrac{3n + 1}{n + 2}$. Find k.

Find also the least value of $n \in \mathbf{N}$ for which

$$k - \frac{3n + 1}{n + 2} < \frac{1}{1000}.$$

(b) A sequence is defined by

$$a_1 = 0, \quad a_n = \sqrt{2 + a_{n-1}}.$$

Prove by induction that

$$a_n \leqslant 2 \text{ for all } n \in \mathbf{N}_0.$$

(c) Use the fact that $(x - 1)^2 \geqslant 0$ for all $x \in \mathbf{R}$, or otherwise, to prove that for $x > 0$

$$x + \frac{1}{x} \geqslant 2.$$

4. (a) Find the value of the derivative with respect to θ of

$$\sqrt{1 + \sin^2\theta}$$

when $\theta = \frac{\pi}{3}$ and express your answer in the form $\frac{1}{a}\sqrt{\frac{b}{c}}$ where $a, b, c \in \mathbf{N}$.

(b) A tangent is drawn to the graph of

$$y = \frac{\sin x}{1 + \tan x}$$

at the point $(0, 0)$. Find the measures of the angles that this tangent makes with the x-axis.

(c) Differentiate $\log_e \dfrac{x^2}{x^2 + 1}$ with respect to x and give your answer in the form $\dfrac{k}{f(x)}$, where k is a constant.

5. If x and y are positive real variables such that

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{4},$$

express y in terms of x and hence find the local minimum value of $x + y$. Prove that there is no value of $x + y$ which is less than this minimum.

6. (a) Evaluate $\int_0^1 (1 + x^2)(1 + x)^2 \, dx$.

(b) Find the value of $\int_{-\frac{4}{3}}^{0} e^{3x + 4} \, dx$ as accurately as the tables for e^x allow.

(c) Evaluate $\int_{-1}^{0} \frac{dx}{x^2 + 2(x + 1)}$.

(d) If $t = \tan \frac{x}{2}$, use your tables, page 9, to express $\sin x$ in terms of t and hence, or otherwise, find

$$\int_{\pi/3}^{\pi/2} \frac{dx}{\sin x}.$$

7. (a) Evaluate $(\sqrt{x+1} + \sqrt{x})(\sqrt{x+1} - \sqrt{x})$ and hence find the sum of the first n terms of the series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1} + \sqrt{n}}.$$

Test the series for convergence.

(b) State the Comparison Test for the convergence of a series of positive terms. Hence, or otherwise, test for convergence the series

$$\frac{1}{\sqrt{2}} + \frac{2}{\sqrt{3}} + \frac{3}{\sqrt{4}} + \cdots + \frac{n}{\sqrt{n+1}} + \cdots$$

(c) Find the range of values of x for which

$$\sum_{n=1}^{\infty} \frac{2^n (\sin x)^n}{n^2}, \quad 0 \leqslant x \leqslant \pi,$$

converges.

8. (a) If E and F are events, illustrate on a Venn diagram that

$$P(E) = P(E \backslash F) + P(E \cap F),$$

where $P(X)$ is the probability of the event X and then prove that

$$E, F \text{ independent} \Rightarrow E, F' \text{ independent.}$$

If $\#(E \backslash F) = 1$, $\#(F \backslash E) = 9$, $\#(E \cap F) = 7$, $\#(S) = 20$, where S is the sample space, investigate if E and F' are independent.

(b) In testing a popular variety of seeds, it is found that a seed germinates with probability $\frac{3}{4}$.
When 300 seeds of a new variety are tested, it is found that 240 germinate. Investigate if the new variety is better than the popular variety at the 5% level of significance.

OR

8. (a) Write down the image of the vector $x_1 \vec{i} + x_2 \vec{j}$ under the axial symmetry in the line $\mathbf{R}(\vec{i} + \vec{j})$ (i.e. the line through the origin of slope 1).

If f is the axial symmetry in the line $\mathbf{R}(\vec{i} + \vec{j})$, investigate if f is a linear transformation.

(b) The point o is the circumcentre of a Δabc. If, taking o as origin, $\vec{k} = \vec{a} + \vec{b} + \vec{c}$, prove that $(\vec{k} - \vec{a}) \perp (\vec{b} - \vec{c})$.

Prove also that $\vec{k}$ is the orthocentre of Δabc.

(c) Let $\vec{a} = 5\vec{i} + 7\vec{j}$ and $\vec{b} = -2\vec{i} + 3\vec{j}$.
If $\vec{r} = \vec{a} + t\vec{b}$ where $t \in \mathbf{R}$, draw the locus of $\vec{r}$.

Let this locus cut the $\vec{i}$-axis at p and the $\vec{j}$-axis at q. Find $\vec{p}$ and $\vec{q}$ in terms of $\vec{i}$ and $\vec{j}$.

r is a point of $[pq]$ such that $|qr| : |rp| = 3 : 2$.
Express $\vec{r}$ in terms of $\vec{i}$ and $\vec{j}$.