AN ROINN OIDEACHAIS M.50

LEAVING CERTIFICATE EXAMINATION, 1979

MATHEMATICS - HIGHER LEVEL - PAPER II (300 marks)

WEDNESDAY, 13 JUNE - MORNING 9.30 to 12.00

Attempt **QUESTION 1** (100 marks) and **FOUR** other questions (50 marks each)

1. (i) Illustrate on the Argand diagram the locus of z such that
$$|z + 3i| = |z - 3|.$$

(ii) £100 is invested at the beginning of each year for 10 consecutive years at 15% per annum compound interest. Express in the form $k(q^n - 1)$ the amount of this investment one year after the final investment was made.

(iii) The sum of the first 21 terms of an arithmetic series is zero. Express in terms of a (i.e. the first term of the series) the sum of the next 21 terms.

(iv) Draw a rough graph of the curve $y^2 = \frac{x}{1-x}$, assuming the y-axis is a tangent to it.

(v) The radius r of a circle is increasing at the rate of 3·5 cm/sec. Find in terms of π the rate at which the area is increasing when $r = 3\frac{1}{7}$ cm.

(vi) If $u_n = 2 + \frac{1}{n}$ is the nth term of a sequence, find the least value of k for which $u_n \leqslant k$ for all n.
Is there a least value of k for which $u_n < k$ for all n? Give your reason.

(vii) Find $\int \sin^2(2x + 1)\, dx + \int \cos^2(2x + 1)\, dx$.

(viii) Test for convergence the series $1 + \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} + \cdots + \frac{1}{n\sqrt{n}} + \cdots$

(ix) Let $x_{r+1} = x_r(2 - 4x_r)$, $r = 1, 2, 3, \cdots$. If $x_1 = 0{\cdot}2$, find x_2 and x_3.

(x) If f is a linear transformation, prove that the points which represent $f(\vec{o})$, $f(\vec{a})$, $f(\vec{b})$, $f(\vec{a} + \vec{b})$ form a parallelogram where o is the origin.

or (x) $\overline{x}$ and σ are the mean and standard deviation of $x_1, x_2, x_3, \cdots, x_n$. Find the mean of
$$\frac{x_1 - \overline{x}}{\sigma}, \frac{x_2 - \overline{x}}{\sigma}, \frac{x_3 - \overline{x}}{\sigma}, \cdots, \frac{x_n - \overline{x}}{\sigma}.$$

2. (a) If $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$ and $rl\, z$ means the real part of z, prove
$$z_1.\overline{z}_2 + \overline{z}_1.z_2 = 2rl\, z_1.\overline{z}_2$$
and illustrate on a diagram $rl\, z \leqslant |z|$.

(b) If $1, z_1, z_2, z_3$ are the roots of $z^4 - 1 = 0$, evaluate $(1 - z_1)(1 - z_2)(1 - z_3)$.

(c) Find the image of the line $|z - 1| = |z - i|$ under the transformation $z \rightarrow i\overline{z}$.

3. Prove by induction that
$$\frac{1}{n!} < \frac{1}{2^{n-1}} \text{ for all } n > 2 \text{ where } n \in \mathbf{N}.$$
The nth term of a sequence is $2 - \frac{1}{2^{n-1}}$. Prove that the sequence is increasing.
Show that
$$1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}$$
is bounded above by 3 (i.e. < 3 for all n) and that the sum of the series
$$1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} + \cdots$$
does not exceed 3.

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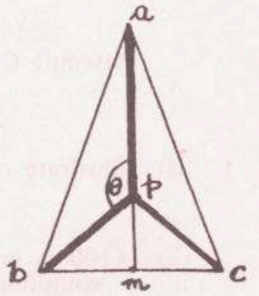
4. (a) Differentiate from first principles the function $x \to \sin x$.

(b) (i) Find the derivative of $x\sqrt{9 - x^2}$ and write your answer in the form $\dfrac{f(x)}{\sqrt{g(x)}}$.

(ii) Find the slope of the tangent at the point (1, 1) to the graph of $y = e^{\left(\frac{1 - x}{1 + x}\right)}$.

(c) If $x = \cos t + t\sin t$ and $y = \sin t - t\cos t$, find $\dfrac{dy}{dx}$ in terms of t.

5. abc is an isosceles triangle, as in diagram, having $|ab| = |ac|$ and $am \perp bc$. Let $|bm| = k$ and $|am| = h$. p is any point in $|am|$ and $|\angle apb| = \theta = |\angle apc|$.



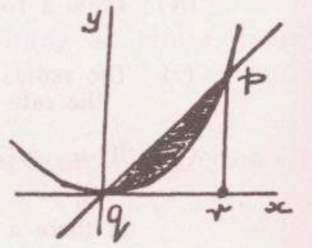
Express $|pb|$ and $|pc|$ in terms of k and θ.
Express $|pa|$ in terms of k, θ, h.

Hence find the minimum value of $|pa| + |pb| + |pc|$ in terms of k and h.
Verify that this minimum is less than $|ma| + |mb| + |mc|$.

6. (a) Evaluate (i) $\displaystyle\int_0^1 x^{2\frac{1}{2}}(\sqrt{x} + 7)\,dx$ (ii) $\displaystyle\int_{-\sqrt{3}}^{\sqrt{3}} \frac{x\,dx}{\sqrt{1 + x^2}}$ (iii) $\displaystyle\int_0^{\pi/2} \frac{\cos\theta\ d\theta}{\sin^2\theta + 4\sin\theta + 5}$.

(b) Let A_n be the area of the region in the first quadrant enclosed between the graphs
$y = x$ and $y = x^{2n}$, $n \in \mathbf{N}_0$, as in diagram.

Prove $\lim\limits_{n\to\infty} A_n$ = area of Δpqr, where $pr \perp qr$.



7. (a) $\dfrac{n^2 + 3n + 6}{3n^2 + 5}$ is the nth term of a sequence. Prove that the sequence is convergent and investigate if the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 + 3n + 6}{3n^2 + 5}$ is convergent.

(b) Test for convergence the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{x^2 + n^2} = \frac{1}{x^2 + 1^2} + \frac{1}{x^2 + 2^2} + \cdots$

(c) Test for convergence or divergence the series $\displaystyle\sum_{n=1}^{\infty} \frac{(n + 1)x^n}{(n + 2)(n + 3)}$ for $x > 0$.

8. (a) If E and F are mutually exclusive events, prove $P(E \cup F) = P(E) + P(F)$, where $P(X)$ denotes the probability of the event X.

(b) x and z are random variables having normal distributions. $\overline{x}$ and 0 are the means of x and z, respectively. σ and 1 are the standard deviations of x and z, respectively. Write down a relation between z and x, $\overline{x}$, σ.

A certain model of car does on average 60 km per gallon of petrol the standard deviation being 10 km. One car is selected at random from a large batch of the same model. Find the probability that the car selected does between 50 and 70 km per gallon of petrol.

(c) A glass rod of unit length falls and breaks into three pieces of lengths x, y, $1 - (x + y)$. Indicate on a diagram all the points (x, y) which satisfy the inequality $x + y < 1$ and find the probability that the three pieces form a triangle.

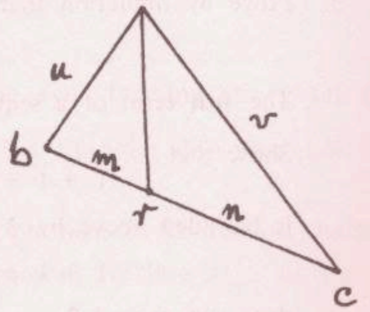
OR 8. (a) If $\vec{x} = -3\vec{i} + 4\vec{j}$ and $\vec{y} = 5\vec{i} + 12\vec{j}$ and $\vec{z} = |\vec{y}|\,\vec{x} + |\vec{x}|\,\vec{y}$, verify that z is on the bisector of the $\angle xoy$, where o is the origin.

(b) If r is a point on the line pq, prove $\vec{r} = t\vec{q} + (1 - t)\,\vec{p}$ for $t \in \mathbf{R}$.

Deduce that if r divides $[pq]$ internally in the ratio $m : n$, then

$$\vec{r} = \frac{m\vec{q} + n\vec{p}}{m + n}.$$

In the Δabc
$|ab| = u$ and $|ac| = v$ and ar bisects the $\angle bac$.



Taking a as origin, express $\vec{r}$ in terms of the unit vector along ab, the unit vector along ac and a scalar t.

If r divides $[bc]$ in the ratio $m : n$, prove $u : v = m : n$.