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LEAVING CERTIFICATE EXAMINATION, 1979

MATHEMATICS–HIGHER LEVEL–PAPER I (300 marks)

MONDAY, 11 JUNE–MORNING, 9.30 to 12.00

Attempt **QUESTION 1** (100 marks) and **FOUR** other questions (50 marks each)

1. (i) Show that every solution x, y, z of the simultaneous equations

$$\begin{aligned} x + y - 2z &= 9 \\ 2x + y - 3z &= 15 \end{aligned}$$

is an arithmetic sequence and find one solution.

(ii) How many natural numbers can be formed by using some or all of the digits 0,1,2,3 if no digit is repeated ?

(iii) Evaluate in the form $k\,2^t$, where $k, t \in \mathbf{N}$,

$$\sum_{r=1}^{100} r\binom{100}{r} = \binom{100}{1} + 2\binom{100}{2} + 3\binom{100}{3} + \cdots\cdots$$

(iv) Find the equation of the line through $(1, 1)$ which also contains the point of intersection of the two lines

$$4x - 7y + 13 = 0 \quad \text{and} \quad 7x + 9y - 17 = 0 .$$

Give your answer in the form $ax + by + c = 0$.

(v) Investigate if the line $x + 3y + 16 = 0$ is a tangent to the circle $x^2 + y^2 - 12x + 2y - 3 = 0$.

(vi) Evaluate $\begin{pmatrix} 2 & -3 \\ 1 & -2 \end{pmatrix}^{10}$.

(vii) The transformation having matrix $\begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}$ maps $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ to $\begin{pmatrix} 0 \\ -5 \end{pmatrix}$. Find the matrix of a transformation which maps $\begin{pmatrix} 0 \\ -5 \end{pmatrix}$ to $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$.

(viii) If $\cos 2A = \frac{12}{13}$, find the possible values of $\tan A$.

(ix) The function $x \to a + b \cos kx$ has period 8π and range $[2, 3]$. Find a, b, k.

(x) If $k \in G, \times$, (i.e. a group under multiplication), prove $(k^{-1})^{-1} = k$, where k^{-1} indicates the inverse of k.

OR (x) Find a condition that the chord joining the two points $(4p^2, 8p)$ and $(4q^2, 8q)$ of the parabola $y^2 = 16x$ contains the focus of the parabola.

2. (a) Find a condition for which the pair of simultaneous equations

$$\begin{aligned} ax + by &= 0 \\ cx + dy &= 0 \end{aligned}$$

has a solution other than the trivial $(0, 0)$.
For what value of t will the simultaneous equations

$$\begin{aligned} 5x + y &= 0 \\ 26x + ty &= 0 \end{aligned}$$

have non-trivial solutions ?

(b) $3 + i$ is a root of the equation

$$x^3 - kx^2 + 22x - 20 = 0.$$

Find the value of $k \in \mathbf{R}$ and the other roots.

(c) If α, β, δ are the roots of the equation

$$x^3 - 5x + 3 = 0,$$

find the equation whose roots are $2\alpha - 1,\ 2\beta - 1,\ 2\delta - 1$.

3. (a) Write down the middle term in the expansion of $(x - 2y)^{12}$ and find its value when $x = 1\frac{1}{2},\ y = \frac{1}{6}$.

(b) If x is so small that its cube and higher powers may be neglected, find an approximation of the form $a + bx + cx^2$ for

$$\sqrt[3]{\frac{1 - 2x}{8(1 + 4x)}} .$$

Hence find, to two places of decimals, the value of $\frac{1}{3}\sqrt[3]{3}$.

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4. Find the radius of the circle that is inscribed in the triangle $a(2, 3)$, $b(-2,-5)$, $c(-4, 6)$.

5. (a) Find the equations of the circles which touch both axes and which contain the point $(3, 6)$.

(b) $h(1, 0)$ and $k(p, q)$ are two fixed points. Find the equation of the locus of a point $t(x, y)$ such that $|\angle htk| = 90°$.
If $q = 4$, find the condition that must be satisfied by p if the locus is not to cut the y-axis.

6. (a) Let $M = \begin{pmatrix} 3 & 4 \\ -2 & -3 \end{pmatrix}$. Find $\lambda_1, \lambda_2 \in \mathbf{R}$ such that

$$M\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \lambda_1\begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \text{and} \quad M\begin{pmatrix} -2 \\ 1 \end{pmatrix} = \lambda_2\begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

If $A = \begin{pmatrix} 1 & -2 \\ -1 & 1 \end{pmatrix}$, evaluate $A^{-1} M A$.

Show that $(A^{-1} MA)^2 = A^{-1} M^2 A$ and hence, or otherwise, evaluate M^{100}.

(b) f is the projection of the plane on the line $y = 2x$ parallel to the line $2y = x$. Find the matrix of f.
A line segment parallel to the y-axis is of length k. The image of this line segment under f is of length h. Find the ratio $h : k$.

7. (a) What is meant by the period of a periodic function ?
If $f(x)$ has period p, prove that $f(mx)$ has period $\frac{p}{m}$ for $m \in \mathbf{R} \setminus \{0\}$.

Write down the range and the period of the function

$$x \rightarrow 3 + 2\sin\frac{x}{2}, \quad x \in \mathbf{R}$$

and indicate roughly the graph of this function for $0 \leqslant x \leqslant 12\pi$.

(b) Use the expansion of $\sin 3x$ in terms of $\sin x$ to verify the following statement:
If f and g are functions of periods h and k, respectively, and $F = f - g$ has period u, then u need not be the l.c.m. of h and k.

8. (a) The following four functions are defined on $\mathbf{R} \setminus \{0\}$.

$$f : x \rightarrow x \qquad g : x \rightarrow \frac{1}{x} \qquad h : x \rightarrow -x \qquad k : x \rightarrow -\frac{1}{x}.$$

Assuming that the composition of functions is associative, prove that $\{f, g, h, k\}$ is a commutative group under composition and write out all its subgroups.

Define what is meant by saying that two groups are isomorphic and show that the above group is isomorphic to the group $\{1, 3, 5, 7, (\text{mod } 8)\}$ under multiplication.

(b) Let $\{e, a, b, c\}$ be a group of order 4 under multiplication, where e is the identity.
If $a^2 = b$, prove that $a^3 \neq a$ and $a^3 \neq b$.
Show that $a^3 = e \Rightarrow ac \neq e$ and $ac \neq b$.

OR 8. (a) Find the equation of the axis of the parabola

$$9x^2 - 6x - 52y + 105 = 0$$

and prove that the parabola does not cut the x-axis.

Find the coordinates of its focus and the equations of its directrix and draw a rough sketch of the curve.

(b) Prove that the line $y = mx + c$ is a tangent to the parabola $y^2 = 4ax$ if $c = \frac{a}{m}$.
m_1 and m_2 are the slopes of the two tangents drawn from a point p to the parabola $y^2 = 4ax$. If $m_1 + m_2 = k$(constant), find the equation of the line on which p lies.

If $k = 2$, find the length of the line segment on which p cannot lie.