AN ROINN OIDEACHAIS

LEAVING CERTIFICATE EXAMINATION, 1978

MATHEMATICS—HIGHER LEVEL—PAPER II (300 marks)

TUESDAY, 13 JUNE—MORNING 9.30 to 12

Attempt **QUESTION 1** (100 marks) and **FOUR** other questions (50 marks each).

1. (i) If z is a complex number such that $z^2 + z + 2 = 0$, what is the value of $\bar{z}(\bar{z} + 1)$?

(ii) What sum of money should be invested now at 10% per annum to give a yearly income of £100 beginning one year from now?

(iii) In an arithmetic series the sum of the first 25 terms is 30. Find the 13th term.

(iv) If $y = x^{10}$, find $\frac{dy}{dx}$ from first principles.

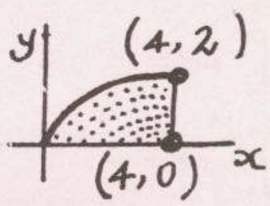
(v) Find the constant of integration if $\int (x^2 + 3x + 1)\, dx = 1$ when $x = \frac{1}{2}$.

(vi) Express $1{\cdot}\dot{2}$ $(= 1{\cdot}222\ldots)$ in the form $\frac{a}{b}$ for $a, b \in \mathbf{N}$.

(vii) Prove by induction that $n(n-1)$ couples (excluding identity couples) can be formed from n points.

(viii) u_n and v_n are the n-th terms of two series and $\lim\limits_{n\to\infty} \frac{u_n}{v_n} = 0$. If $v_n = n$, find a possible u_n such that $\sum\limits_{n=1}^{\infty} u_n$ is convergent.

(ix) Calculate the area of the region enclosed by the parabola and the line segments as indicated in the diagram.



(x) If $\vec{x} = \frac{\sqrt{3}}{2}\,\vec{\imath} + \frac{1}{2}\,\vec{\jmath}$, find a unit vector $\vec{y} = y_1\,\vec{\imath} + y_2\,\vec{\jmath}$, where $y_1 \neq 0$, such that $\vec{x}\cdot\vec{y} = \frac{1}{2}$.

OR

(x) Use the tables on page 36 to find the area under the normal curve which corresponds to $-2{\cdot}17 \leqslant z \leqslant 2{\cdot}17$.

2. (*a*) If z_1 and z_2 are complex numbers, prove that

$$\overline{z_1 \cdot z_2} = \bar{z}_1 \cdot \bar{z}_2$$

If $z = \frac{z_1}{z_2}$, prove that $\bar{z} = \frac{\bar{z}_1}{\bar{z}_2}$

(*b*) Illustrate the set A of solutions of $|z - 4i| \leqslant 5$, where z is a complex number.

If B is the set of solutions of $|z - 6 + 4i| \leqslant k$, find the greatest value of $k \in \mathbf{N}$ for which $A \cap B = \phi$.

If f is the isometry $z \to -\bar{z}$, find $f(A)$.

3. (*a*) Test for convergence the series

$$\sum_{n=1}^{\infty} \frac{3n}{2^{n+1}} = \frac{3.1}{2^{1+1}} + \frac{3.2}{2^{2+1}} + \ldots$$

Hence, or otherwise, find $\lim\limits_{n\to\infty} \frac{n}{2^n}$ giving a reason for your answer.

(*b*) Test for convergence the series

$$\sum_{n=1}^{\infty} \frac{b^n}{n(a+n-1)}\; x^n$$

where a, b are positive constants and $x > 0$.

If x must also satisfy the inequality $\frac{x}{b} \leqslant 2$, find the maximum value of x for which the series converges.

[**P.T.O.**

4. (*a*) Find the $\lim_{x\to\infty} \frac{x-1}{x+2}$ and write down the equations of the asymptotes of the curve $y = \frac{x-1}{x+2}$ for $x \in \mathbf{R}$, $x \neq -2$. Draw a rough sketch of the curve.

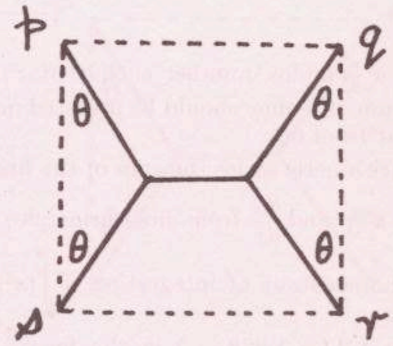
(*b*) Differentiate with respect to x:

(i) $\left(\frac{x^2-1}{x^2+1}\right)^3$ (ii) $\frac{2\sin 3x}{\sec 2x}$

(iii) $\frac{x\,e^{1+x}}{e^{1-x}}$ (iv) $x^{\log x}$

5. p, q, r, s are four houses at the vertices of a square of side $2k$ metres in length. The line segments represent a water pipe system connecting the four houses and $0 < \theta < \frac{\pi}{2}$, see diagram. Express the length of the pipe system in terms of θ and k and hence find the minimum length of piping required.

Verify that this length is less than the sum of the two diagonals (i.e. $|pr| + |qs|$).



6. Find:

(i) $\int \frac{x(x+1)}{x^4}\,dx$ (ii) $\int_0^1 x\sin(x^2-1)\,dx$

(iii) $\int_0^{\pi/4} \sin^3 2x \cos 2x\,dx$ (iv) $\int_1^2 \frac{dx}{\sqrt{(1+x)(3-x)}}$

(v) $\int_0^{\pi/2} \cos^2 x \,.\, e^x \,.\, e^{\left(\frac{\sin 2x}{2}\right)}\,dx$

7. (*a*) A glass rod of unit length falls and breaks into three pieces of lengths x, y and $1-(x+y)$. If these three pieces are to form a triangle, show that $x + y \geqslant \frac{1}{2}$ and find $u \in \mathbf{R}$ for which $x \leqslant u$.

(*b*) Given $\epsilon > 0$ and $n > 1$, find a value of k in terms of ϵ for which

$$\frac{2n^2+n-3}{n^2-1} - 2 < \epsilon$$

for all $n > k$.

8. (*a*) If E and F are events, prove $P(E \cup F) = P(E) + P(F) - P(E \cap F)$, where $P(X)$ denotes the probability of the event X.

A multiple choice test consists of 10 questions. To each question there are three possible answers denoted by (*a*), (*b*), (*c*) and these answers are chosen at random by the candidates.

Let E be the event: (*a*) is the answer chosen for each of the *first* four questions.

Let F be the event: (*a*) is the answer chosen for each of the *last* four questions.

Find the total number of possible answers and calculate $P(E \cup F)$.

(*b*) A die is thrown 125 times and a "four" comes up k times. Between what limits must k lie in order to be able to conclude that at the 5% level of significance the die is *not* biased?

OR

8. (*a*) Define a linear transformation. If u and v are linear transformations, prove that their composite uv is also linear.

Let p be the point which represents the vector $2\vec{\imath} + 3\vec{\jmath}$ and let f be the central symmetry in p. If $\vec{x} = \vec{\imath} + \vec{\jmath}$ and $\vec{y} = 6\vec{\imath} - \vec{\jmath}$, find $f(\vec{x})$, $f(\vec{y})$ and $f(\vec{x}+\vec{y})$ in terms of $\vec{\imath}$ and $\vec{\jmath}$ and investigate if f is linear.

(*b*) Prove that in any triangle, the circumcentre, the orthocentre and the centroid are collinear.

$a\,b\,c$ is an isosceles triangle having centroid at g and orthocentre at k. If $|ag| = 50/3$ and $|ak| = 24$, find the length of the base $[bc]$.