LEAVING CERTIFICATE EXAMINATION, 1978

MATHEMATICS—HIGHER LEVEL—PAPER I (300 marks)

THURSDAY, 8 JUNE—MORNING, 9.45 to 12.15

Attempt **QUESTION 1** (100 marks) and **FOUR** other questions (50 marks each)

1. (i) If $2x - 1$ is a factor of $2x^3 + tx^2 + 2x + t$, find t.

(ii) Find the term of the binomial expansion of $\left(y^2 - \frac{1}{y}\right)^{30}$ which does not contain y.

(iii) How many arrangements can be made from the letters e, e, e, r, w, z taking them three at a time?

(iv) Find the equations of the lines which are parallel to the line $3x + 4y + 5 = 0$ and which are four units from the origin.

(v) Tangents are drawn to the circle $x^2 + y^2 = 25$ at the points (4, 3) and (–3, 4). Find the measure of the angle between them.

(vi) Evaluate $\begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix}^{-1} \begin{pmatrix} 3 & 1 \\ -4 & 8 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix}$.

(vii) Write down the image of the vector $\begin{pmatrix} x \\ y \end{pmatrix}$ under the linear transformation $\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ and then say what is the geometrical meaning of the transformation.

(viii) If $\tan \frac{A}{2} = \frac{1}{5}$, $0 \leqslant A \leqslant \frac{\pi}{2}$, evaluate $\sin 2A$, without finding the value of A.

(ix) Prove: $\sin A + \sin B = 2 \sin \frac{A + B}{2} \cos \frac{A - B}{2}$.

(x) If G is a group under multiplication and if $p, r \in G$ such that $p^{-1} r^2 p = rp$, express r in terms of p explaining each step.

OR—

(x) Find the coordinates of the vertex of the parabola $y^2 + 4x - 6y + 13 = 0$.

2. (a) Given the simultaneous equations

$$x + y + z = 1$$
$$2x - 2y + tz = 4$$
$$3x - y + 2z = 6$$

find the value of t for which there is no solution.

(b) Find, correct to one place of decimals, a real root of the equation $x^3 + x^2 - 1 = 0$.
Prove that the equation has only one real root.

3. (a) Simplify $\left(1 + \frac{a}{b}\right)^4 + \left(1 - \frac{a}{b}\right)^4$ and hence, or otherwise, write

$$(x + \sqrt{x^2 - 1})^4 + (x - \sqrt{x^2 - 1})^4$$

as a polynomial in x.
Evaluate $(8 + \sqrt{65})^4 + (8 - \sqrt{65})^4$

(b) Write out the first four terms of the expansion of $(1 - x)^{-1/2}$ and hence, or otherwise, find, correct to four places of decimals, the value of $\frac{1}{7}\sqrt{50}$.

4. Let K be the line $ax + by + c = 0$, where $c > 0$. What is the distance of the origin from K ?

Investigate if one or both of the two points (20, 20), (– 22, – 20) are on the same side of the line $20x - 21y + 15 = 0$ as the origin.

Find the equation of the bisector of that angle between the two lines
$12x^2 + 25xy + 12y^2 - 12x + 12y - 144 = 0$
which does not contain the origin.

5. Prove that the two circles

$$S_1 : x^2 + y^2 + 4y + 3 = 0$$
$$S_2 : x^2 + y^2 - 12x - 12y - 49 = 0$$

touch each other and find the equation of their common tangent (T).

Find the equation of a circle that has T as its common tangent with S_1 and which touches the x-axis.

6. (a) T is the triangle formed by the three vectors $\begin{pmatrix}0\\0\end{pmatrix}$, $\begin{pmatrix}1\\1\end{pmatrix}$, $\begin{pmatrix}1\\4\end{pmatrix}$ and $\begin{pmatrix}3 & 1\\-4 & 8\end{pmatrix}$ is the matrix of a linear transformation f. Find the image of T under f and evaluate the ratio area of T : area of $f(T)$.

(b) r is a rotation about the origin of angle $\theta < \frac{\pi}{2}$ which maps $\begin{pmatrix}1\\1\end{pmatrix} \to \frac{1}{2}\begin{pmatrix}\sqrt{3}-1\\\sqrt{3}+1\end{pmatrix}$. Find the matrix of r.

g is a transformation such that the composite gr is the axial symmetry in the line $y = x$. Find the matrix for g and say what is the geometrical meaning of g.

7. (a) If $x \neq n\pi$ for $n \in \mathbf{Z}$, find one value of x which satisfies

$$\sin 3x + \sin 4x = \sin x.$$

(b) Write down the period of the function

$$f : \mathbf{R} \to \mathbf{R} : x \to \sin 3x.$$

If $g : \mathbf{R} \to \mathbf{R} : x \to \sin kx$ has a period $\frac{3\pi}{4}$, find k.

Find the period of $f(x) + g(x)$.

Show by using $\sin 2x$ and $\sin \pi x$ that the sum of two periodic functions is NOT always a periodic function.

Find a sine function that has period 3 and range [1, 3].

8. (a) The following three functions are defined on $\mathbf{R}\setminus\{0, 1\}$:

$$f : x \to x$$
$$g : x \to \frac{1}{1-x}$$
$$h : x \to \frac{x-1}{x}.$$

Prove that the set $\{f, g, h\}$ is a group under composition.

(b) Let G be a group under multiplication and let H be a subset of G. If $xy^{-1} \in H$ for all $x, y \in H$, prove

(i) $e \in H$, where e is the identity in G

(ii) $x^{-1} \in H$,

(iii) $xy \in H$.

Deduce that H is a subgroup of G.

OR

8. (a) P is a locus such that each point of P is the same distance from (1, 1) as it is from the line $x - 2y + 2 = 0$. Find where P cuts the x-axis and prove that it does not cut the y-axis. Draw a rough sketch of P.

(b) Verify that $h(at_1^2, 2at_1)$ and $k(at_2^2, 2at_2)$ are points of the parabola $y^2 = 4ax$.

If the line hk contains the focus of the parabola, find a relation between t_1 and t_2.

h and k are any two points of the parabola $y^2 = 4ax$ such that the line hk contains the focus. The line through h parallel to the x-axis meets the line through k parallel to the y-axis at r. Find the equation of the locus of r in terms of x, y and find where this locus cuts the parabola.