LEAVING CERTIFICATE EXAMINATION, 1977

MATHEMATICS—HIGHER LEVEL—PAPER I (300 marks)

MONDAY, 13 JUNE—MORNING, 9.30 to 12.

Six questions to be answered.
All questions carry equal marks.
Mathematics Tables may be obtained from the Superintendent.

1. Prove that the distance of the point (x_1, y_1) from the line $ax + by + c = 0$ is given by

$$\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}},$$

where (x_1, y_1) is on the same side of the line as the origin and $c > 0$. Deduce the equations of the bisectors of the angles between the line

$$3x + 4y + 8 = 0 \text{ and the } x\text{-axis.}$$

A triangle is formed by the three lines

$$y = 0; \quad 3x + 4y + 8 = 0; \quad 4x + 3y + 3 = 0.$$

Verify that one vertex of this triangle is in the 4-th quadrant (i.e. $x > 0$, $y < 0$). A circle is inscribed in the triangle. Calculate the length of its radius.

2. Find the equations of the two circles which have the x-axis as tangent and which pass through the two points $(2, -2)$ and $(-5, -1)$.
If the circles touch the x-axis at p and q, calculate $|pq|$.

3. (*a*) The equation of a parabola is given as

$$y(y-6) = 3x.$$

Find
(i) the coordinates of its vertex and of its focus
(ii) the equation of its axis and of its directrix.

Sketch the parabola and by considering the area of a certain triangle, or otherwise, prove that the area of the region enclosed by the y-axis and the parabola is greater than 9.

(*b*) Find the equations of the two tangents to the ellipse

$$\frac{x^2}{4} + y^2 = 1$$

which are parallel to the line $x = y$.

4. (*a*) f is a linear transformation such that

$$f(\vec{\imath}) = 4\vec{\imath} + 2\vec{\jmath} \text{ and } f(\vec{\jmath}) = -3\vec{\imath} - \vec{\jmath},$$

where $\vec{\imath}$ and $\vec{\jmath}$ are perpendicular unit vectors. Write down the matrix for f and hence, or otherwise, find the image of the vector $\vec{\imath} + \vec{\jmath}$ under f.
The vectors

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

form a triangle pqr whose image under f is the triangle xyz. Calculate the ratio:

$$\text{area of } \triangle xyz : \text{area of } \triangle pqr.$$

(*b*) Let

$$M = \begin{pmatrix} -3 & -2 \\ 8 & 7 \end{pmatrix}.$$

Let λ_1, λ_2 be the roots of the quadratic equation $\lambda^2 - 4\lambda - 5 = 0$ and let $\lambda_1 > \lambda_2$.
(i) Find two non-null vectors

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \text{ and } \vec{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

for which

$$M\vec{x} = \lambda_1 \vec{x} \text{ and } M\vec{y} = \lambda_2 \vec{y}.$$

(ii) Verify that $M^2 - 4M - 5 = 0$.

5. (*a*) Write down two linear factors of $x^2 + y^2$ and hence, or otherwise, find two factors of 5 whose sum is 2.
Factorise

$$x^2 - 2x + 5$$

and verify your answer.

(*b*) Prove
(i) $|z|^2 = z \cdot \bar{z}$ where z is a complex number
(ii) $\overline{z_1 \cdot z_2} = \bar{z}_1 \cdot \bar{z}_2$ where $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$

and deduce that

$$\overline{z_1 \cdot z_2 \cdot z_3} = \bar{z}_1 \cdot \bar{z}_2 \cdot \bar{z}_3$$

Let $f(z) = a_0z^3 + a_1z^2 + a_2z + a_3$, where a_0, a_1, a_2, a_3 are real. Given that z_1 is a root of $f(z) = 0$, prove that $\bar{z}_1$ is also a root.
If $(2 + 3i)$ is a root of $3z^3 - 13z^2 + 43z - 13 = 0$, find the other two roots.

[P.T.O.

6. What is meant by saying a function is periodic? Write down the period and range of

(i) $x \to \sqrt{2} \sin \dfrac{x}{\sqrt{2}}$ (ii) $x \to -\sin(x+2)$,

where both functions are defined on $\mathbf{R}$.
If the function $\mathbf{R} \to \mathbf{R} : x \to a + b \sin kx$ has period 2 and range $[0, 1]$, find the values of the positive real numbers a, b, k.
If the function $\mathbf{R} \to \mathbf{R} : x \to \sin 3x + \sin tx$ has period 6π, find one value of $t \in \mathbf{R}$.

7. (*a*) What is meant by the centroid of a triangle? $opqr$ is a parallelogram and m is the midpoint of $[or]$. The line pm cuts the line oq at k. Taking o as the origin, express the vector $\vec{k}$ in terms of $\vec{p}$ and $\vec{r}$ and find the ratio $|\vec{ok}| : |\vec{kq}|$.

(*b*) Define the scalar product of two vectors. Let f be the scalar product function defined by

$$f(\vec{x}, \vec{y}) = \vec{x} \cdot \vec{y}.$$

Prove

(i) $f(a\vec{x}, \vec{y}) = af(\vec{x}, \vec{y})$, for $a > 0$

(ii) $f(\vec{x}_1 + \vec{x}_2, \vec{y}) = f(\vec{x}_1, \vec{y}) + f(\vec{x}_2, \vec{y})$

What is meant by saying that f is a bilinear function (i.e. a linear function in two variables)?

8. (*a*) Prove by induction that

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i \sin n\theta \qquad \text{for } n \in \mathbf{N}.$$

Show that the above result is also true for $n \in \mathbf{Z}$.

(*b*) If $t = \tan\dfrac{A}{2}$, prove that $\sin A = \dfrac{2t}{1+t^2}$.

Use your Tables to find a value of A in the domain $\pi/2 < A < \pi$ which satisfies the equation

$$3 \sin A - 2\cos A = 3.$$

Verify that $A = \pi/2$ is also a solution.

9. Find the matrix with respect to perpendicular unit vectors of the anti-clockwise rotation R_θ about the origin which maps the vector

$$\begin{pmatrix} 3 \\ 4 \end{pmatrix} \text{ to the vector } \begin{pmatrix} -4 \\ 3 \end{pmatrix}.$$

If S_o is the central symmetry in the origin, find the equation of the image of the curve $y^2 = x(2-x)$ under S_o after R_θ.

10. A. (*a*) Let H be the set of all numbers of the form

$$p - q\sqrt{2} \qquad \text{for } p, q \in \mathbf{Z}.$$

Prove that

(i) H is closed under addition

(ii) addition is associative in H

(iii) there is an identity element for addition in H

(iv) for every element in H under addition there is an inverse element also in H.

If "addition" is replaced by "multiplication" in (i), (iii), (iv), above, which of them, if any, is now false? Give your reasons.

(*b*) $pqrs$ is a rectangle, not a square. Find two lines X, Y and a point c such that $pqrs$ maps onto itself under S_X, S_Y, S_c, where these are respectively the axial symmetry in X and in Y and the central symmetry in c. Draw up a Cayley table for the set $\{1_\pi, S_X, S_Y, S_c\}$ under composition where 1_π is the identity transformation, and investigate if it has all the group properties.
Let $\{p, q, r, s\}$ be the set of images of $\{p, q, r, s\}$ under a transformation. Write down four couples of this transformation which is not the result of 1_π, S_X, S_Y, S_c or their composition.

OR

10. B. (*a*) There are 12 red, 6 blue and 2 yellow marbles in a jar. A marble is taken at random from the jar, its colour is recorded and it is put back again. This is done 5 times in all. Find the probability that the red colour is recorded

(i) exactly twice

(ii) at most twice

(iii) at least three times.

The experiment of drawing 5 marbles from the jar, as described above, is performed 5 times and a person bets that in at least one of the experiments the red colour will appear exactly twice. Investigate if this bet has a good chance of winning.

(*b*) A piece of glass is in the shape of a triangle and x, y, z are the midpoints of its sides, as in diagram. A speck of dust falls at random on the glass. Find the probability that the speck falls on the triangular piece.