AN ROINN OIDEACHAIS

LEAVING CERTIFICATE EXAMINATION, 1976

MATHEMATICS—HIGHER LEVEL—PAPER II (300 marks)

MONDAY, 14 JUNE—MORNING 9.30 to 12

Six questions to be answered.
All questions carry equal marks.
Mathematics Tables may be obtained from the Superintendent.

1. Verify that $x^3 - x^2 + 1 = 0$ has a root between 0 and -1 and find this root correct to *one* place of decimals.

   Let

$$A(x) = \int_k^x (3t^2 - 2t)\, dt,$$

   where k is a constant.

   Find k, correct to one place of decimals, if $A(1) = 1$.

2. (*a*) If $(r + \frac{1}{2})^3 - (r - \frac{1}{2})^3 = ar^2 + b$ is true for all values of r, where a, b are independent of r, find the value of a and the value of b. Hence, or otherwise, find an expression for

$$1^2 + 2^2 + \ldots + n^2.$$

   (*b*) Use the binomial theorem to find the first three terms in the expansion of $\sqrt{1+x}$ and of $\sqrt{1-x}$. Hence, or otherwise, find, in ascending powers of x, the first three terms in the expansion of

$$\sqrt{\frac{1+x}{1-x}}$$

   and use this expansion to evaluate

$$\sqrt{\frac{1{,}001}{999}},$$

   correct to five places of decimals.

3. (*a*) The sum to n terms of a series is given by

$$S_n = n^2 + n + 1.$$

   Write down an expression for T_n, the n-th term, for $n > 1$ and investigate if the series is arithmetic, geometric or neither.

   (*b*) If $k > 0$, prove that

$$\sqrt{\frac{1{,}001}{999}},$$

correct to five places of decimals.

3. (*a*) The sum to n terms of a series is given by

$$S_n = n^2 + n + 1.$$

Write down an expression for T_n, the n-th term, for $n > 1$ and investigate if the series is arithmetic, geometric or neither.

(*b*) If $k > 0$, prove that

$$(1 + k)^n > \frac{n(n-1)\ldots(n-r)}{1.\,2.\ldots(r+1)}\,k^{r+1} \qquad (n > r)$$

and deduce that

$$\frac{n}{(1+k)^n} < \frac{(r+1)!}{k^{r+1}\,n^r}\,\frac{1}{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{r}{n}\right)}$$

Hence, or otherwise, find

$$\lim_{n\to\infty}\frac{n}{p^n} \text{ for } p > 1,$$

noting that r and k are fixed constants.
Evaluate

$$\lim_{n\to\infty}\frac{2^n}{n}.$$

4. (*a*) A function f is defined for different intervals of its domain $\mathbf{R}$ as follows:

$$\begin{aligned} f(x) &= x && \text{for } -\infty < x \leqslant 0 \\ f(x) &= x + 1 && \text{for } 0 < x \leqslant 1 \\ f(x) &= x - 1 && \text{for } x > 1. \end{aligned}$$

Sketch the graph of the function. Write out the values of $f(0)$, $f(1)$, $f(2)$ and find the values of x for which $f(x) = 1\frac{1}{2}$.

(*b*) (i) Let S_n be the sum of the first n terms of the series of positive terms $u_1 + u_2 + \ldots + u_r + \ldots$. Write u_n as a difference of two sums and show that if the series converges,

$$\lim_{n\to\infty} u_n = 0.$$

Test for convergence the series

$$\frac{1+3(1^2)}{1+1^2} + \frac{1+3(2^2)}{1+2^2} + \ldots + \frac{1+3(n^2)}{1+n^2} + \ldots$$

(ii) Test for convergence

$$\sum_{r=1}^{\infty}\frac{2^r}{r^2} = \frac{2}{1^2} + \frac{2^2}{2^2} + \frac{2^3}{3^2} + \ldots$$

[P.T.O.

5. (*a*) Assuming

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1,$$

differentiate from first principles $\cos \theta$ with respect to θ.

(*b*) Differentiate with respect to x

(i) $\cos^2 3x$ (ii) $\log(1 + \tan^2 x)$ (iii) $\sqrt{\frac{1+x}{1-x}}, x \neq 1.$

If $x = ke^{-t/2}$, where k is independent of t, find the value of k if

$$\frac{dx}{dt} = 2e \text{ when } t = -2.$$

6. The function

$$f : x \to \frac{1}{1 + e^{-x}}$$

is defined for all $x \in \mathbf{R}$.

(i) Find the range of f.
(ii) Show that f is an increasing function.
(iii) Show that f has a point of inflexion at $x = 0$ and find the gradient of the tangent to f at this point.
(iv) Draw a rough sketch of the function to illustrate (i), (ii), (iii) above.

7. The shape of a playing field is a rectangle with semicircular ends and its complete boundary is to be used as a running track 400 m in length. If the rectangular region is to have maximum area, find the total length of the semicircular ends.

8. Evaluate

(i) $\int_0^1 e^2 \, dt$ (ii) $\int_{\frac{\pi^2}{9}}^{\frac{\pi^2}{4}} \frac{\cos \sqrt{x} \, dx}{\sqrt{x}}$ (iii) $\int_1^2 \log x \, dx$ (iv) $\int_0^{\pi/2} \frac{\cos^3\theta \, d\theta}{1 + \sin^2 \theta}$

9. Let V_1 be the volume generated when

$$y^2 = x \quad , 0 \leqslant x \leqslant 1$$

is rotated about the x-axis.

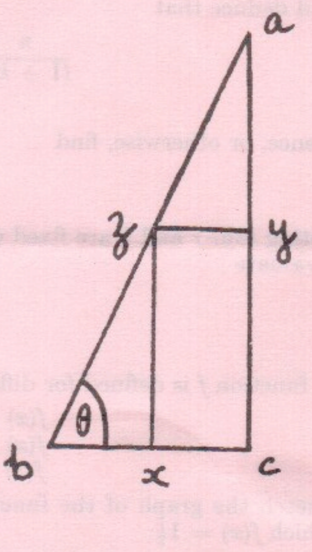
Let V_2 be the volume generated when

$$y^2 = x, \quad 0 \leqslant x \leqslant 1$$

is rotated about the line $x = 1$.

Investigate if $2V_1 = V_2$.

10. abc is a triangle in which $\angle acb$ is a right angle and in which $\tan \theta = \sqrt{5}$. x, y, z are points in the sides as in diagram such that $xz \parallel ac$, $zy \parallel bc$.



Let $| bx | = q$, $| xz | = t_1$, $| zy | = p$, $| ya | = t_2$.

If $t_1 > p$ prove that $t_2 < 5q$.

$$\left(\text{Use } \sqrt{5} = \frac{t_1}{q} = \frac{t_2}{p} \right).$$

Verify that

(i) $p + t_2 < t_1 + t_2 < t_1 + 5q$
(ii) $p + t_2 < p + 5q < t_1 + 5q$

and say why

$$\frac{p + 5q}{p + q}$$

can be taken as an approximation for $\sqrt{5}$.

Taking

$$\frac{p}{q} = \frac{1}{1}$$

as a first approximation for $\sqrt{5}$, use the iteration (formula)

$$\frac{p}{q} \to \frac{p + 5q}{p + q}$$

to write down five further approximations for $\sqrt{5}$.