AN ROINN OIDEACHAIS

LEAVING CERTIFICATE EXAMINATION, 1976

MATHEMATICS—HIGHER LEVEL—PAPER I
(300 marks)

THURSDAY, 10 JUNE—MORNING, 9.30 to 12.

Six questions to be answered.
All questions carry equal marks.
Mathematics Tables may be obtained from the Superintendent.

1. (*a*) Sketch the lines $y = m_1 x + 4$ where $c_1 = 1$, $m_1 \in \{0, 1, 2, 3\}$ and the lines $y = m_2 x + c_2$ where $m_2 = 1$, $c_2 \in \{0, 1, 2, 3\}$. Find the measure of the acute angle between $y = 4x + 4$ and one of the lines $y = m_2 x + c_2$.

(*b*) p, q, r are the points $(-2, 1)$, $(2, 3)$, $(3, 1)$, respectively. What is the equation of the locus

$$\{k \mid \triangle kpq = 2 \triangle kpr \text{ in area}\}?$$

2. (*a*) Find the equations of the circles which touch the x-axis at the point $(4, 0)$ and which cut the y-axis at points which are 6 units apart.

(*b*) Sketch on your answer book the curve

$$\left(\frac{x}{2}\right)^2 - y^2 = 1.$$

Verify that $(5/2, 3/4)$ is a point of the curve and find the equation of the tangent to the curve at this point.

3. $p(x_1, y_1)$ and $q(x_2, y_2)$ are two points of the parabola $y^2 = 4x$ such that $(4, 2)$ is the mid-point of $[pq]$. Prove that

$$(y_1 - y_2) = (x_1 - x_2)$$

and hence write down the equation of the line pq.
Calculate $|pq|$.

4. (*a*) If

$$X = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} \text{ and } Y = \begin{pmatrix} -1 & 1 \\ 0 & -2 \end{pmatrix}$$

show that $(X + Y)^2 \neq X^2 + 2XY + Y^2$.

(*b*) What is meant by the statement

$$\text{“}\begin{pmatrix} -1 & 2 \\ -3 & 4 \end{pmatrix} \text{ is the matrix of a linear transformation”?}$$

Find λ_1, λ_2 for which

$$\begin{pmatrix} -1 & 2 \\ -3 & 4 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \lambda_1 \begin{pmatrix} 2 \\ 3 \end{pmatrix} \text{ and } \begin{pmatrix} -1 & 2 \\ -3 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \lambda_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Express the vector

$$\begin{pmatrix} 3 \\ 5 \end{pmatrix}$$

in the form

$$t\begin{pmatrix} 2 \\ 3 \end{pmatrix} + k\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

where $t, k \in \mathbf{R}$ and hence, or otherwise, find the image of the vector

$$\begin{pmatrix} 3 \\ 5 \end{pmatrix}$$

under the linear transformation

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} -1 & 2 \\ -3 & 4 \end{pmatrix}^{10} \begin{pmatrix} x \\ y \end{pmatrix}$$

5. If $z = \cos\theta + i \sin\theta$, show that

$$\frac{1}{z} = \cos\theta - i \sin\theta \text{ and } \frac{1}{z^2} = \cos 2\theta - i \sin 2\theta.$$

Show that

$$\left(z + \frac{1}{z}\right)^4 = 16 \cos^4\theta$$

and hence, or otherwise, show that

$$\cos^4\theta = \tfrac{1}{8}(\cos 4\theta + 4 \cos 2\theta + 3).$$

Express $\cos 4\theta$ as a polynomial in $\cos\theta$.

6. (*a*) What is meant by saying that a function is periodic?
Write down the period of each of the following two functions defined on $\mathbf{R}$:

$$x \to \sin 3x \text{ and } x \to \cos 5x.$$

What is the period of the function

$$x \to \sin 3x + \cos 5x?$$

(*b*) l is the period of a function

$$f : \mathbf{R} \to \mathbf{R} : x \to f(x).$$

If $f(0) = 1$, find $f(3l)$.
k is the period of a function

$$g : \mathbf{R} \to \mathbf{R} : x \to g(x).$$

If l and k are prime numbers, find the period of each of the functions

$$\text{(i)} \quad x \to f(x) + g(x), \qquad \text{(ii)} \quad x \to \frac{f(x)}{g(x)},\ g(x) \neq 0.$$

7. Define the scalar product (dot product) of two vectors and show that for the non-null vectors $\vec{x}$ and $\vec{y}$

(i) $\vec{x} \cdot \vec{x} = |\vec{x}|^2$

(ii) $\vec{x} \cdot \vec{y} = 0 \Rightarrow \vec{x} \perp \vec{y}$

(iii) $\vec{x} \cdot \vec{y} = \vec{y} \cdot \vec{x}$.

The circumcentre of a $\triangle abc$ is o. Taking o as the origin, prove that

$$(\vec{a} + \vec{b}) \perp (\vec{a} - \vec{b}).$$

If $k = \vec{a} + \vec{b} + \vec{c}$, prove that k is the orthocentre of the $\triangle\ abc$.

8. With respect to the $\vec{i}, \vec{j}$ basis write down the matrix for

(i) a reflection, P, in a line passing through the origin and making an angle measuring $\pi/3$ with $\mathbf{R}\vec{i}$.

(ii) a rotation, Q, about the origin of angle measuring $\pi/3$.

Write down the matrix for $P \circ Q$ and say what single transformation is equal to $P \circ Q$.
Find the equation of the image of the line $y + \sqrt{3}\,x = 4$ under the transformation $P \circ Q$.

9. (*a*) Express $\sqrt{-5 + 12i}$ in the form $a + ib$ for $a, b \in \mathbf{R}$ and $i = \sqrt{-1}$.

(*b*) Express $\sin 3\theta$ as a polynomial in $\sin \theta$ and hence write down an expression for $\sin^3\theta$.

Use the substitution $x = y \sin \theta$, $y > 0$, in the equation $x^3 - 12x + 8 = 0$ and so write down another expression for $\sin^3\theta$. Hence find one root of $x^3 - 12x + 8 = 0$ as accurately as the Tables allow.

10. A. (*a*) Prove that the set of matrices

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$$

where $a \in Z_5$ (i.e. the set of residue classes mod 5) is a commutative group under multiplication.

(*b*) Let the operation * be defined on $\mathbf{R}_0 \times \mathbf{R}$ by

$$(p, q) * (r, s) = (pr, ps + q),$$

where $\mathbf{R}_0$ is the set of real numbers excluding zero. Write down the identity for * and the inverse of (a, b). Is the set $\mathbf{R}_0 \times \mathbf{R}$ a group under *? Give your reason.

OR

10. B. (*a*) Three dice are thrown at once. Write down the probabilities of getting exactly 3 sixes, 2 sixes, one six, no six.

Three dice are thrown 432 times. A person bets that exactly two sixes will appear 25 times or more. Do you think this person has a good chance of winning the bet? Give your reason.

(*b*) A new cure for a certain sickness in animals is to be tested. 100 sick animals are chosen for the experiment. The new cure is given to some and the old cure is given to the others. The following table gives the results of the experiment:

	New Cure	Old Cure
Improvement	30 animals	20 animals
No Improvement	10 animals	40 animals

Let K be the event: an animal is given the new cure.
Let T be the event: an animal improves.
Do the figures in the table show that the events K and T are independent? Give a reason for your answer.