AN ROINN OIDEACHAIS

LEAVING CERTIFICATE EXAMINATION, 1975

MATHEMATICS—HIGHER LEVEL—PAPER II (300 marks)

MONDAY, 16 JUNE—MORNING 9.30 to 12

Six questions to be answered.
All questions carry equal marks.
Mathematics Tables may be obtained from the Superintendent.

1. (*a*) If α, β and γ are the roots of the equation $10x^3 - 17x^2 + x + 7 = 0$, find the equations having roots
   (i) $-\alpha, -\beta, -\gamma$,
   (ii) $10\alpha, 10\beta, 10\gamma$.

   (*b*) Show that the equation $10x^3 - 17x^2 + x + 7 = 0$ has a root between -1 and 0 and find this root correct to 2 places of decimals.

2. (*a*) How many different permutations can be made out of all the letters taken together of the word EXAMINATION?
   In how many of these permutations will
   (i) two N's be adjacent
   (ii) all the vowels be together?

   (*b*) Write down and simplify the first four terms of the binomial $(1 + x)^n$ in ascending powers of x when $n = 1/3$.
   Hence, evaluate, correct to six places of decimals, $\sqrt[3]{8{\cdot}04}$.

3. (*a*) Given that $-1 < r < 1$ and that

$$S_n = \sum_{t=1}^{n} r^t \,; \qquad \left[\sum_{t=1}^{n} r^t = r + r^2 + r^3 + \ldots\right],$$

   show that

$$\lim_{n\to\infty} S_n = \frac{r}{1-r}$$

   and find the value of

$$\sum_{j=1}^{n} S_j . \qquad \left[\sum_{j=1}^{n} S_j = S_1 + S_2 + S_3 + \ldots\right]$$

   (*b*) Find the sum of the series

$$\sum_{t=1}^{n} tr^t$$

   and using the *fact* that

$$\lim_{n\to\infty} \frac{n}{p^n} = 0 \quad \text{for } p > 1,$$

   deduce that

$$\sum_{t=1}^{\infty} tr^t = \frac{r}{(1-r)^2}\ , \qquad \text{where } -1 < r < 1.$$

4. (*a*) Prove that the series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \ldots + \frac{1}{n^p} + \ldots$$

   is *not convergent* for $p = 1$.
   Show that

$$\frac{1}{n^p} > \frac{1}{n} \quad \text{for } p < 1,\ n > 1,$$

   and hence, or otherwise, prove that the series is also *not convergent* for $p < 1$.

   Prove that the series is *convergent* for $p > 1$.

   (*b*) Examine for convergence the series:

   (i) $\dfrac{1.3}{1!} + \dfrac{2.5}{2!} + \dfrac{3.7}{3!} + \ldots + \dfrac{n(2n+1)}{n!} + \ldots$

   (ii) $\dfrac{1}{1.3}\dfrac{x}{2} + \dfrac{1}{2.5}\dfrac{x^2}{2^2} + \dfrac{1}{3.7}\dfrac{x^3}{2^3} + \ldots + \dfrac{x^n}{n(2n+1)2^n} + \ldots$

5. (a) Assuming that $$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1,$$

evaluate (i) $\lim_{x \to 0} \left(\frac{\sin x/2}{x} \right)$ (ii) $\lim_{x \to \pi} \left(\frac{\cos x/2}{\pi - x} \right)$

(b) Differentiate with respect to x:

(i) $\frac{(1+x)^2}{1+x^2}$ (ii) $\sec x \tan x$ (iii) x^x for $x > 0$.

(c) If $y = (\sin^{-1} x)^2$, show that
$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 2.$$

6. (a) The line $y = (r/h)\,x$ is rotated about the x-axis to form a right circular cone of base radius r and vertical height h. Prove by integration that the volume of this cone is $\frac{1}{3}\pi r^2 h$.

(b) Sand falls on a horizontal floor at the rate of 9 m^3 per minute and forms a heap in the shape of a right circular cone with vertical angle 60°. Find the rate at which the height of the heap is increasing 1 minute after the sand begins to fall. (Take 3 as an approximate value for π).

7. f is a function defined as follows: $f : x \to \sin x - x \cos x = f(x),\ x \in R$.

(i) Evaluate $f(-\pi)$, $f(0)$, $f(\pi)$.

(ii) Prove that f is an increasing function in the domain $-\pi < x < \pi$.

(iii) Find the maximum and minimum points of f in the domain $-\pi \leqslant x \leqslant \pi$ and show there is a point of inflexion at $x = 0$. Prove also that f has a point of inflexion between $x = 116°$ and $x = 117°$.

(iv) Sketch the graph of f in the domain $-\pi \leqslant x \leqslant \pi$.

8. (a) Evaluate:

(i) $\int_1^4 \frac{x^2 - 1}{\sqrt{x}}\,dx$ (ii) $\int_0^{\pi/4} \sin x \ \cos 3x\,dx$ (iii) $\int_e^{e^2} \frac{dx}{x \log_e x}$.

(b) Prove that $\frac{1}{x^2} \leqslant \frac{1}{x} \leqslant \frac{1}{\sqrt{x}}$ for $x \geqslant 1$.

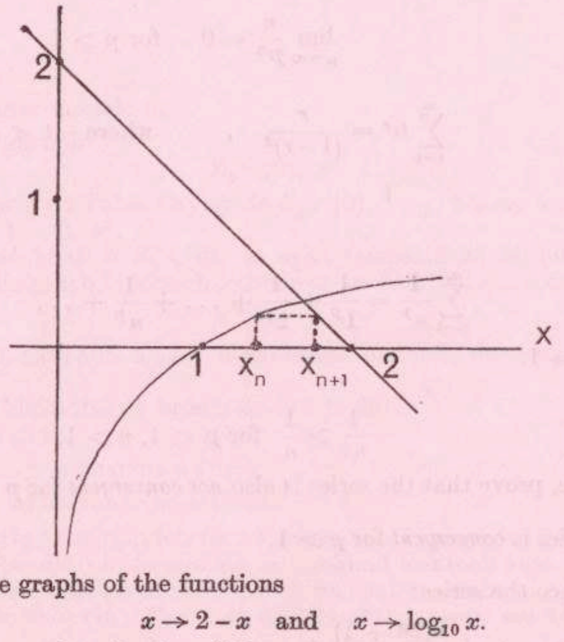
By integrating between suitable values, show that
$$\tfrac{1}{2} \leqslant \log_e 2 \leqslant 2\sqrt{2} - 2 .$$

9. Find the area of the smaller region enclosed by

the ellipse $$\frac{x^2}{4} + y^2 = 1$$ and

the parabola $$y^2 = \frac{3x}{4}.$$

10.



The diagrams shows the graphs of the functions
$$x \to 2 - x \quad \text{and} \quad x \to \log_{10} x.$$

Find from the diagram a value of x for which
$$2 - x = \log_{10} x, \text{ approximately.}$$

If the ordinates at x_n and at x_{n+1} are equal, as in diagram, write down an equation in x_n and x_{n+1}. Use this equation to find two further values of x which approximately satisfy
$$2 - x = \log_{10} x .$$