L 37 **AN ROINN OIDEACHAIS** M. 49

LEAVING CERTIFICATE EXAMINATION, 1975

MATHEMATICS—HIGHER LEVEL—PAPER I

(300 marks)

WEDNESDAY, 11 JUNE—MORNING, 9.45 to 12.15

Six questions to be answered.
All questions carry equal marks.
Mathematics Tables may be obtained from the Superintendent.

1. (*a*) If $3x + 2y + m = 0$ and $5x - y + n = 0$ are the equations of the two lines
$$15x^2 + 7xy - 2y^2 - 26x + ky + 8 = 0, \quad m, n, k \in Z$$
find the values m, n and k.

(*b*) Find the point of intersection of the lines

(i) $x = \dfrac{2t}{t+1}, y = \dfrac{2-t}{t+1}$

(ii) $x = \dfrac{t}{t+1}, y = \dfrac{3t-2}{t+1}, \quad t \in R \backslash \{-1\}$.

2. If $S_1 = 0$ is $x^2 + y^2 - 4x + 8y - 30 = 0$ and $S_2 = 0$ is $x^2 + y^2 + 8x - 16y + 30 = 0$, prove that $S_1 + \lambda S_2 = 0$, $\lambda \neq -1$, is the equation of a circle through the points of intersection of S_1 and S_2.
Find the equation of the circle whose diameter is the join of the points of intersection of $S_1 = 0$ and $S_2 = 0$.
When $\lambda = -1$, what locus is represented by $S_1 + \lambda S_2 = 0$?

3. (*a*) Find the vertex and length of the latus rectum of each of the following parabolas

(i) $(y-3)^2 = -12(x+2)$

(ii) $x^2 - 2x + 2y + 6 = 0$.

Make a rough sketch of each parabola indicating the axis in each case.

(*b*) Show that the two tangents from (1, 0) to the parabola $x^2 - 2x + 2y + 6 = 0$ are $y = \pm\sqrt{5}\,(x-1)$

(*c*) $[pq]$ is a chord through the focus of the parabola $y^2 = 4ax$ and the coordinates of p are $(at^2, 2at)$. Find the coordinates of q in terms of t.

4. Let
$$M = \begin{bmatrix} 9 & -2 \\ 8 & -1 \end{bmatrix}$$
be the matrix of a linear transformation and let $\begin{bmatrix} u \\ v \end{bmatrix}$ be the image of the vector $\begin{bmatrix} x \\ y \end{bmatrix}$ by this transformation.

Express x in terms of u and v.

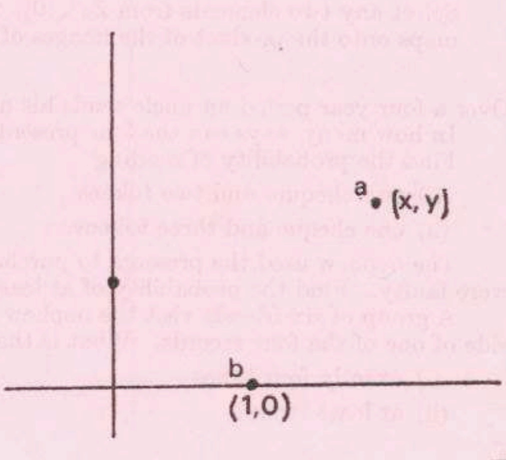
If $\begin{bmatrix} x \\ y \end{bmatrix}$ is a vector such that $M\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$, find values of a, b, c, d for which
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Hence find any three non-null vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ for which $M\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$.

If $\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$, write the equation $y^2 - 5xy + 4x^2 = 0$ in terms of α and β.

5. (*a*) When $z_1 = 2 + 3i$, $z_2 = 3 + 2i$ calculate
(i) $z_1 z_2$ (ii) $\overline{z_1 z_2}$ (iii) $\bar{z}_1 \bar{z}_2$.

(*b*) The points a, b on the diagram have the co-ordinates (x, y), $(1, 0)$. Find the distance $|ab|$ and show that this distance is equal to the modulus $|z - 1|$, when $z = x + iy$.
Illustrate on an Argand diagram the locus of z for which
$$1 \leqslant |z + 2| \leqslant 2.$$



[P.T.O.

6. (*a*) Sketch the portion for $-2\pi \leqslant x \leqslant 2\pi$ of the graph of each of the functions:

(i) $f : x \to \cos x + |\cos x|$

(ii) $g : x \to \cos x - |\cos x|$

(iii) $h : x \to -2\,|\cos x|$

and from the graphs write down the period and range of each of the functions *f*, *g* and *h*.

(*b*) Find the period and range of the function

$$x \to 3 \cos 3x + 4 \sin 3x.$$

7. (*a*) Show that the greatest angle of the $\triangle\, abc$, where $\vec{a}$ is $2\vec{j}$, $\vec{b}$ is $4\vec{i} + 2\vec{j}$ and $\vec{c}$ is $\vec{i} - \vec{j}$, is

$$\cos^{-1} \frac{1}{\sqrt{10}}.$$

(*b*) Prove that the image of the vector $\vec{v}$ by orthogonal projection on the vector $\vec{x}$ is

$$p(\vec{v}) = \frac{\vec{v} \cdot \vec{x}}{|\vec{x}| \cdot |\vec{x}|}\,\vec{x}$$

Hence, or otherwise, find the image of $\vec{b}$ on $\overrightarrow{ac}$, by orthogonal projection, where $\vec{a}, \vec{b}, \vec{c}$ are the vectors as in (*a*) above.

8. (*a*) One root of $z^3 - az^2 + bz - 30 = 0$, $a, b \in R$ and $z \in C$ is $1 + 3i$. Find the values of *a* and *b* and the other roots of the equation.

(*b*) The function $x \to \tan^{-1} x$ is defined for $x \in R$ and for principal values

$$-\frac{\pi}{2} \text{ to } \frac{\pi}{2} \text{ of } \tan^{-1} x.$$

(i) Draw a rough graph of the function and explain the shape of the graph as $x \to \pm\infty$.

(ii) Write down the range of the function.

(iii) Use the tables to find the approximate values of $\tan^{-1} 4$ and $\tan^{-1}(-3)$.
Hence, say whether or not

$$\tan^{-1} x - \tan^{-1} y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$$

is true for $x = 4$, $y = -3$.

9. (*a*) If *f* denotes reflection in $R_i^{\to}$ (i.e. the *x*-axis), prove that

$$f(\vec{a} + \vec{b}) = f(\vec{a}) + f(\vec{b}),$$
$$f(\lambda\vec{a}) = \lambda f(\vec{a})$$

for all vectors $\vec{a}$ and $\vec{b}$ and all real numbers λ.
What terminology can be applied to *f* because of these properties?.

(*b*) Write down the matrix for reflection in a line which makes an angle θ with $R_i^{\to}$
Hence, find the matrix for reflection in the line

$$y - 2x = 0$$

and calculate the image of the vector $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ by this reflection.

10A. (*a*) Draw up a Cayley Table for

(i) $Z_5, +$

(ii) $Z_5, \times$

(iii) $\{1, i, -1, -i\}, \times$. (Z_5 is the set of residue classes modulo 5)

(*b*) Arrange the elements of the group

$$Z_5 \setminus \{0\}, \times$$

so that a Cayley table of $Z_5 \setminus \{0\}, \times$ shows on inspection the isomorphism of this group and the group $\{1, i, -1, -i\}, \times$.
Select any two elements from $Z_5 \setminus \{0\}, \times$ and show that the image, in $\{1, i, -1, -i\}$, of their product maps onto the product of the images of the two elements selected.

Or

10B. Over a four year period an uncle sends his nephew a yearly present of *either* a cheque or a gift token.
In how many ways can the four presents be chosen by the uncle?
Find the probability of sending

(i) two cheques and two tokens

(ii) one cheque and three tokens.

The nephew used the presents to purchase four records, all bought at a shop where 2% of the records were faulty. Find the probability of at least one faulty record.

A group of six friends visit the nephew and during the visit, each of the friends plays, at random, a side of one of the four records. What is the probability that any one side be played

(i) exactly four times

(ii) at least twice?