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LEAVING CERTIFICATE EXAMINATION, 1974

MATHEMATICS—HIGHER LEVEL—PAPER I
(300 marks)

MONDAY, 17th JUNE—MORNING, 9.30 to 12

Six questions to be answered.
All questions carry equal marks.
Mathematics Tables may be obtained from the Superintendent.

1. Establish that an angle θ, $\theta < \frac{\pi}{2}$, between two intersecting straight lines of gradients m_1 and m_2 is given by

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2}.$$

$p(0, -3)$ and $r(5, 0)$ are two vertices of a square $pqrs$ such that $[pr]$ is a diagonal. Find the gradients of ps and pq. Hence, or otherwise, calculate the coordinates of s and q.

2. Prove that when the two circles $x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ and $x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$ intersect at right angles $2g_1g_2 + 2f_1f_2 = c_1 + c_2$.
$a(1, 1)$ and $b(2, 3)$ are points on a plane. Show that the set of points S, where $S = \{s \mid |as| = 2|sb|\}$ is a circle.
Find the equation of a second circle, centre $(0, 0)$, which intersects S orthogonally.

3. The equation of a parabola is $x = 5y - 2x^2$. Find
   (i) the coordinates of the focus,
   (ii) the equation of the directrix.
Hence sketch the parabola.
Show that $x = 2t$, $y = \pm 3\sqrt{1-t^2}$ is the parametric equation of an ellipse.

4. $\begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}$ is the matrix of a transformation, f, with respect to the orthonormal vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ as basis. Illustrate in a diagram the images of the vectors $\begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ by the transformation f and verify that f is linear but not isometric.

If λ_1, λ_2 are the roots of the quadratic equation $\lambda^2 - 7\lambda + 10 = 0$, find any two non null vectors $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ for which

$$\begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \lambda_1 \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \text{ and } \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \lambda_2 \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$$

where λ_1 is the greater of the two roots.
Find the values of a, b, c, d for which

$$\begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Verify that

$$\begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{pmatrix}\begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 0 & 2 \end{pmatrix}.$$

5. (*a*) If $\bar{z}$ is the complex conjugate of the non-zero complex number z, find two arguments of z for which $z - \left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\bar{z} = 0$.

(*b*) If $z = x + iy$ express (i) $|z|$, (ii) $|z-3|$, (iii) $|z+1|$ in terms of x and y.
Solve the equation, $|z-3| = |z+1|$ and indicate on the Argand diagram the set

$$K = \{z \mid |z-3| = |z+1|\}.$$

Choose one element of K and for this element verify that $|z-3| = |z+1|$.

6. (*a*) Sketch the graphs of the functions

(i) $x \to \cos\ |x|$ and

(ii) $x \to |\cos x|$ in the domain $-2\pi \leqslant x \leqslant 2\pi$

and from the graphs write down the period and range of (i) and (ii).

Find the period and range of each of the functions

(iii) $x \to \cos^2 x$,

(iv) $x \to \sin^2 x \cos^2 x$

and sketch the graphs of (iii) and (iv) in the domain $-2\pi \leqslant x \leqslant 2\pi$.

(*b*) Prove that $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\dfrac{x+y}{1-xy}$, $|xy| < 1$ and hence find the value of

$$\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{8}.$$

7. (*a*) If $\vec{r}_1 = 3\vec{i} + 4\vec{j}$, $\vec{r}_2 = -5\vec{i} + 12\vec{j}$, find (i) $|\vec{r}_1|$ (ii) $\vec{r}_1 . \vec{r}_2$ (iii) the angle θ between $\vec{r}_1$ and $\vec{r}_2$ to the nearest degree.
Find the locus of $\vec{r}$ such that $(\vec{r} + \vec{r}_1) \perp (\vec{r} - \vec{r}_1)$ where $\vec{r} = x\vec{i} + y\vec{j}$. Illustrate your answer by a diagram.

(*b*) The vectors $\vec{r}$ and $\vec{s}$ are shown in the diagram.
Copy the diagram into your answer book and indicate on it the meaning of

$$\frac{(\vec{r} - \vec{s}) . \vec{r}}{|\vec{r}|}.$$

Simplify $\dfrac{\vec{r} . \vec{s}}{|\vec{r}|} + \dfrac{(\vec{r} - \vec{s}) . \vec{r}}{|\vec{r}|}$

8. (*a*) $\vec{p}$ and $\vec{q}$ are the position vectors $(-2, 3)$, $(3, 4)$ respectively. Find the position vectors of

(i) $\overrightarrow{pq}$ (ii) $\overrightarrow{qp}$ and illustrate them on a diagram.

(*b*) Assuming that the axes are perpendicular, find the matrix for

(i) a rotation r through an angle θ with the origin as centre, $\theta < \pi/2$,

(ii) a projection p parallel to the x-axis on the line $y + x = 0$,

(iii) $p \circ r$ (i.e. p after r).

If the image of $(4, 0)$ under $p \circ r$ is $(-2\sqrt{3}, 2\sqrt{3})$, find the value of θ.

9. (*a*) Assuming $\cos 3\theta + i \sin 3\theta = (\cos\theta + i\sin\theta)^3$, show that

$$\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}.$$

(*b*) If $z = 1 + 2i$ is a root of the equation $z^3 + az + b = 0$, where $a, b \in R$, find the values of a, b. Find the other roots of the equation.

10A. (*a*) T is the set of complex numbers of the form $a + b\sqrt{-5}$, where $a, b \in Q$, and a, b are not both simultaneously zero. Show that T is a group under multiplication of complex numbers, assuming associativity of multiplication in T.

(*b*) $A = \{x_1, x_2, x_3, \ldots\}$, $B = \{y_1, y_2, y_3, \ldots\}$. A is a group under the operation $*$ and B is a group under the operation $\oplus$. Define what is meant by saying that there is an isomorphism between the two groups $A, *$ and $B, \oplus$.

$$\text{Let } S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$$

Construct a Cayley table for this set under matrix multiplication. Assuming associativity of matrix multiplication, show from the table, or otherwise, that $S, \times$ is a group. Establish that the group $S, \times$ is isomorphic to the group $Z_4, +$ (i.e. the group of residue classes under addition, mod 4.).

OR

10B. (*a*) There are twenty raffle tickets, each marked with one of the numbers one to twenty. No two tickets bear the same number. One ticket is drawn at random. Find the probability that (i) it is a multiple of 5 or 7 (ii) it is a multiple of 3 or 5.

(*b*) In five throws of a pair of dice what is the probability of throwing doubles (i.e. the same number on each of the two dice) at least twice?