AN ROINN OIDEACHAIS

LEAVING CERTIFICATE EXAMINATION, 1973

MATHEMATICS—HIGHER LEVEL—PAPER II (300 marks)

WEDNESDAY, 13 JUNE—MORNING 9.30 to 12

Six questions to be answered.
All questions are of equal value.
Mathematics Tables may be obtained from the Superintendent.

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1. (*a*) Describe the general appearance and properties of the typical graphs that arise in connection with the solutions of cubic equations with real coefficients when
   (i) all the roots are real and different,
   (ii) all the roots are real but two are equal,
   (iii) all the roots are real and equal,
   (iv) one of the roots is real and the other roots are complex.

   (*b*) One root of the equation
$$2x^3 - x^2 + 2x - 1 = 0$$
is i, where $i = \sqrt{-1}$. Find the other roots of the equation.

2. (*a*) Write down the general term in the expansion of $(1 + x)^n$, where n is a positive integer and by integration, or otherwise, express
$$\sum_{r=0}^{n} \frac{1}{r+1}\ {}^nC_r$$
in terms of n.

   (*b*) If x^4 and higher powers of x can be neglected, simplify
$$\sqrt[3]{1+x} - \left[1 + \frac{1}{3}x - \frac{1}{9}x^2\right]$$
and hence evaluate $\sqrt[3]{1010}$ to five places of decimals.

3. (*a*) Evaluate (i) $\lim\limits_{n\to\infty} \dfrac{n}{2n+1}$, (ii) $\lim\limits_{n\to 0} \dfrac{2n - 3n^2}{5 - 4n + 7n^2}$.
   Find the sum of the series
$$\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \ldots.$$
   Hence, using the result in (i) above, or otherwise, show that this series is convergent.

   (*b*) Show that the series
$$\sum_{r=1}^{\infty} \frac{1}{r+1}$$
is not convergent.
Prove that
$$\frac{1}{\sqrt{r(r+1)}} > \frac{1}{r+1}$$
for all $r \geqslant 1$ and examine for convergence the series
$$\sum_{r=1}^{\infty} \frac{1}{\sqrt{r(r+1)}}\ .$$

4. (*a*) Use the Comparison Test to test the series
$$\sum_{n=1}^{\infty} \frac{1+n}{3 + 2n^2 + n^3}$$
for convergence.

   (*b*) Use the Ratio Test to examine for convergence the series
$$\sum_{r=1}^{\infty} \frac{x^r}{r^2}$$

5. (a) Differentiate from first principles the function $f : x \to \tan x$ with respect to x in the domain

$$-\frac{\pi}{2} < x < \frac{\pi}{2}.$$

Differentiate with respect to x *any three* of the following

(i) $\dfrac{x-1}{x+1}$, (ii) $\sqrt{\dfrac{x-1}{x+1}}$, (iii) $e^{\tan 2x}$, (iv) $x^2 \log_e x$

(b) Show that the derivative of $\sin^{-1} t$ with respect to t is

$$\frac{1}{\sqrt{1-t^2}} \text{ for } |t| < 1.$$

If $x = t + \sin^{-1} t$, prove that

$$(1-t^2)\frac{d^2x}{dt^2} - t\frac{dx}{dt} + t = 0.$$

6. The sum of the surface areas of a sphere and a cube is a constant. Show that the sum of the volumes of the sphere and the cube is least when the length of the edge of the cube is twice the length of the radius of the sphere and find this minimum value.

7. f and g are functions defined as follows:

$$f : x \to \sin x - x\cos x = f(x), \quad x \in R,$$
$$g : x \to x \sin x = g(x), \quad x \in R.$$

(i) Write down the values of $f(\pi)$, $g(\pi)$, $f(-\pi)$, $g(-\pi)$.

(ii) Show that

$$f(x)\,[g(x) - g(-x)] = g(x)\,[f(x) + f(-x)].$$

(iii) Prove that f is an increasing function in the domain $0 < x < \pi$.

(iv) Evaluate

$$\int_0^{\pi/2} f(x)\,dx.$$

8. (a) Evaluate

(i) $\displaystyle\int_1^3 (x^{-3} + x^3)\,dx,$

(ii) $\displaystyle\int_0^{\pi/4} \sin x \sin 3x\,dx,$

(iii) $\displaystyle\int_0^1 \frac{x\,dx}{4 - 4x + x^2}.$

(b) If a is a constant and f is a function such that

$$f(a-x) = f(a) - f(x)$$

for all $x \in R$, show that

$$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$

and hence deduce that

$$\int_0^a f(x)\,dx = \tfrac{1}{2}af(a)$$

If $f : x \to \sin^5 x$, verify that

$$f(2\pi - x) = f(2\pi) - f(x)$$

and hence, or otherwise, evaluate

$$\int_0^{2\pi} \sin^5 x\,dx.$$

9. Show that the graph of

$$y^2 = \frac{1}{1-x^2}$$

has no real points of inflexion. Sketch the graph and indicate the region bounded by the graph and the line $y = 2$.

Calculate the volume generated by rotating this region about the y-axis.

10. Given that x_r is an approximate value of $1/t$, where $t > 0$ and $t \in R$, you can assume that there is a number h_r such that

$$\frac{1}{t} = x_r + h_r.$$

If h_r is so small that its square and higher powers may be neglected, show, by first deducing the approximation for h_r, that

$$x_r(2 - tx_r)[= x_{r+1} \text{ (say)}]$$

is a better approximation for $1/t$.

Given that $0{\cdot}6(= x_0)$ is an approximate value of $1/\sqrt{3}$, find a better approximate value (x_1) using the above formula for x_{r+1}.