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PAPER I

1. Show that the lines $x - 3y + 1 = 0$, $2x + y - 3 = 0$ and $5x - 8y = 0$ are concurrent.
   Prove that the points $(t + 2, t + 2)$, $(t, t + 3)$ and $(t - 2, t)$ are not collinear. Find the images of the points under the translation $(t + 2, t + 2) \to (0, 0)$ and hence find the perimeter and the area of the triangle determined by the three image points.

2. A circle contains the points $(2, -1)$ and $(1, 1)$ and has its centre on the line $y - 3x + 7 = 0$.
   Find:
   (i) the equation of the circle,
   (ii) the points on the circle at which the tangents are parallel to the line $y - 3x + 7 = 0$.

3. (*a*) Find the equation of the parabola with its focus at $(1, 2)$ and directrix the line $3x - 4y + 10 = 0$.
   (*b*) Find the equation of the tangent at $(2, -2)$ to the ellipse
$$\frac{(x-1)^2}{2} + \frac{y^2}{8} = 1$$

4. (*a*) Given the following matrices
$$A = \begin{bmatrix} \frac{1}{2} & \frac{2}{3} \\ \frac{1}{2} & -\frac{2}{3} \end{bmatrix} \quad X = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad Y = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad Z = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad W = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$
   express A in the form $\alpha X + \beta Y + \gamma Z + \delta W$, where $\alpha, \beta, \gamma, \delta$ are real numbers.
   (*b*) If λ_1, λ_2 are the roots of the quadratic equation $\lambda^2 - 3\lambda - 4 = 0$, where λ_1 is the positive root, show that
$$\begin{bmatrix} 2-\lambda_1 & 3 \\ 2 & 1-\lambda_1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$
   Show also that
$$\begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \lambda_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

5. If $\vec{i}$ and $\vec{j}$ are unit vectors along the x-axis and y-axis, respectively, prove that the vector $\vec{p} = 3\vec{i} - 2\vec{j}$ is orthogonal to the line $3x - 2y + 4 = 0$.

   Find a vector $\vec{v}$ and a scalar k such that any point on the line $3x - 2y + 4 = 0$ can be written as $\vec{v} + t(\vec{v} - k\vec{p})$, where $t \in R$.

6. (i) What is meant by the scalar product $\vec{x} \cdot \vec{y}$ of the vector $\vec{x}$ and the vector $\vec{y}$?

   Represent geometrically the equality $\vec{x} \cdot \vec{y} = \vec{y} \cdot \vec{x}$.

   (ii) $\vec{u}, \vec{v}$, and $\vec{w}$ are vectors such that $\vec{u} \perp \vec{w}$ and $\vec{v} \perp \vec{w}$. Prove that $(\vec{u} + \vec{v}) \perp \vec{w}$.
   (Note: the symbol $\perp$ means "is perpendicular to").

   (iii) $\vec{a}$ and $\vec{c}$ are the position vectors $(3, 4)$ and $(5, 1)$, respectively. The vector $\vec{a}$ can be regarded as the sum of two components, one of which is parallel to $\vec{c}$ and the other perpendicular to $\vec{c}$. Find the component which is parallel to $\vec{c}$.
   Find, also, a scalar k such that
$$\vec{a} \cdot \left(\vec{c} - \frac{k\vec{a}}{|\vec{a}|}\right) = 0.$$

7. (i) Prove that
$$\sin 2A = \frac{2 \tan A}{1 + \tan^2 A}$$
and hence, or otherwise, express $\tan 15°$ in surd form.
   (ii) Express $a \cos \phi + b \sin \phi$ in the form $k \cos (\phi - \alpha)$. Find the domain of values of p for which the equation $a \cos \phi + b \sin \phi = p$ has a solution. Hence, or otherwise, solve the equation
$$\cos \theta + \sqrt{3} \sin \theta = 1.$$

8. Prove De Moivres Theorem.
If $z = \cos\theta + i\sin\theta$, prove $z^n + z^{-n} = 2\cos n\theta$ and find $\sin n\theta$ in terms of z.
Prove also that

$$(\sin x + i\cos x)^n = \cos n\left(\frac{\pi}{2} - x\right) + i\sin n\left(\frac{\pi}{2} - x\right),$$

where n is a positive integer.

9. (a) Express $\frac{2+i}{1+i}$ in the form $a + ib$, where a and b are rational.
(b) If $\arg(z_3 - z_2) = \arg(z_2 - z_1)$, prove that z_1, z_2 and z_3 are collinear.
(c) Show that $z = i$ is a solution of $z^2 - z(1+2i) - (1-i) = 0$ and find the other solution of the equation.

10A. Prove that the set K of all numbers of the form $a + b\sqrt{3}$ (a and b integers) is a group under addition. Is K a group under multiplication? Give your reason.
Let $R^* = \{r_1, r_2, r_3\}$ be the set of rotations of the equilateral triangle abc onto itself, and let ρ indicate the reflection in the axis of symmetry containing a. Let $S = \{\rho \circ r_1, \rho \circ r_2, \rho \circ r_3\}$. Is $S, \circ$ a group? Give your reasons.

or

10B. Find the probability of 4 turning up at least once in (i) two tosses, (ii) three tosses of a fair die.
A factory finds that on average 20% of the buttons produced by a given machine will be defective. If 10 buttons are selected at random find the probability
(i) that exactly three of them will be defective,
(ii) that either three or four of them will be defective.