LEAVING CERTIFICATE EXAMINATION, 1972

MATHEMATICS—HIGHER LEVEL—PAPER II (300 marks)

TUESDAY, 13th JUNE—MORNING 9.30 to 12

Six questions to be answered.
All questions are of equal value.
Mathematics Tables may be obtained from the Superintendent.

1. The equation $12x^3 - 8x^2 - x + 1 = 0$ has two equal roots. Find all the roots of the equation and illustrate the main features of the graph of the function $x \to 12x^3 - 8x^2 - x + 1$.

2. (*a*) 12 players take part in a chess competition so that each player plays one game against each of the other players. How many games are played altogether?
(*b*) Prove that

$$(a + x)^n = \sum_{r=0}^{n} {}^nC_r\, x^r\, a^{n-r},\ n \in N_0.$$

Write out the first 4 terms of the binomial expansion

$$(1 + 2x)^{2/3}.$$

If x is so small that its square and higher powers may be neglected, find an approximation of the form $a + bx$ for

$$\frac{\sqrt[3]{(1 + 2x)^2}}{4 - x}.$$

3. For what domain of values of p is the series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

convergent?
Assuming that $n! \sim n^n e^{-n} \sqrt{2\pi n}$ for large values of n, show that

$${}^{2n}C_n\, p^n\, q^n \sim \frac{(4pq)^n}{\sqrt{\pi n}}\ (= T_n)$$

and deduce that

$$\sum_{n=1}^{\infty} T_n$$

is not convergent for $p = \frac{1}{2} = q$.
If $0 < p < 1, 0 < q < 1, p + q = 1, p \neq q$, prove that $4pq < 1$. Put $4pq = t$ and then use the ratio test to test the series

$$\sum_{n=1}^{\infty} T_n$$

for convergence.
[Note: $\sim$ means "is approximately equal to"].

4. Given that

$$0 < q < 1, \qquad p + q = 1, \qquad f_i = p\,(q^{i-1}), \qquad f(x) = \sum_{i=1}^{\infty} f_i\, x^i, \qquad |x| < 1,$$

prove that

$$f(x) = \frac{px}{1 - qx}.$$

If

$$g(x) = \frac{1}{1 - f(x)},$$

prove that

$$g(x) = 1 + \sum_{i=1}^{\infty} px^i.$$

5. Differentiate from first principles the function

$$f : x \to \sqrt{x - 1},\ (x \geqslant 1)$$

with respect to x.
Write down the image (range) of f and say if the function has or has not a turning point.
Draw a graph of the function.
If g is the function defined by

$$g : x \to f'(x) \left(= \frac{df(x)}{dx}\right),$$

write down the domain and image of g and find the value of x for which

$$f(x) = g(x).$$

Investigate whether the tangents to the graphs of f and g at their point of intersection are perpendicular or not.

6. (a) Differentiate with respect to x:

$$\text{(i)}\ \sqrt{1+x^2},\quad \text{(ii)}\ \sin^{-1} x/4,\quad \text{(iii)}\ e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2},\quad \text{(iv)}\ \log_e \cos^2 x.$$

(b) If $y = Ax\,e^{-2x}$, prove that

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 0.$$

If $y = Be^{mx}$, $B \neq 0$, is also a solution of this equation, find the value of m.

7. When a car is travelling at a speed of x m.p.h., a minimum stopping distance for the car is assumed to be $(x + x^2/20)$ feet. A steady stream of cars travels eastwards along a narrow road past a point p on the road at a speed of x m.p.h., each car being 20 feet long. At a fixed time, t, the front of one of the cars is abreast of p and 30 minutes later the front of another car is also abreast of p. If the cars are the assumed minimum stopping distance from one another, calculate the maximum number of cars that could have passed p between t and $t + 30$ minutes.

$$\left[x \text{ m.p.h.} = \frac{22x}{15} \text{ feet per second}\right]$$

8. (a) Evaluate

$$\text{(i)}\ \int_0^{\pi} 2x^3\,dx \qquad \text{(ii)}\ \int_0^{\sqrt{\pi}} x \sin(x^2)\,dx \qquad \text{(iii)}\ \int_0^1 \frac{x^3\,dx}{1+x^2} \qquad \text{(iv)}\ \int_{\frac{1}{2}}^1 x \log_e x\,dx.$$

(b) Use the approximation

$$x - \frac{x^2}{2} + \frac{x^3}{3} \text{ for } \log_e (1+x)$$

to find the approximate value of

$$\int_1^2 \log_e t\,dt.$$

9. Using the same axes and the same scales sketch the graphs of

$$\text{(i)}\ x^2 = 3 - y^2 \qquad \text{(ii)}\ x^2 = 2y$$

and indicate the smaller region enclosed by the two graphs.

Calculate the volume generated by rotating this smaller region about the y-axis.

10. Let x_r be an approximate value of $\sqrt[n]{k}$, $n \in N_0$, $k \geqslant 0$, so that

$$x_r + h = \sqrt[n]{k}.$$

If h is so small that its square and higher powers may be neglected, show that an approximate value of h (say h_r) is given by

$$h_r = \frac{k - x_r^n}{n x_r^{n-1}}$$

and deduce that

$$x_{r+1} (= x_r + h_r) = x_r + \frac{1}{n}\left[\frac{k}{x_r^{n-1}} - x_r\right].$$

Given that $x_1 = 1{\cdot}5$ is the first approximate value of $\sqrt[3]{3}$, find x_2, its second approximate value, using the above formula.