PAPER II

Six questions to be answered. (300 marks)

1. (a) If α, β, γ are roots of the equation
$x^3 - 2x + 5 = 0$
form the equations whose roots are
(i) $\alpha + 1$, $\beta + 1$, $\gamma + 1$;
(ii) 2α, 2β, 2γ.

(b) Show that $x^3 + 2x - 4 = 0$ has a root between $1{\cdot}1$ and $1{\cdot}2$, and find the value of that root correct to 2 places of decimals.

2. (a) Prove that $(x + 1)^n = \sum_{r=0}^{n} {}^nC_r x^r$. (Note: ${}^nC_0 = 1$).

(b) Given that the binomial expansion of $(1 + x)^k$ for k a positive integer also holds true for any rational k provided $|x| < 1$ write down the fourth term of this expansion if $k = \frac{1}{3}$ and $x = \frac{1}{20}$.

(c) Use the binomial theorem to evaluate

$$\sqrt[3]{28}$$

correct to 3 places of decimals.

3. (a) Test the following series for convergence

$$\frac{1}{1!} + \frac{3}{2!} + \frac{5}{3!} + \frac{7}{4!} + \ldots$$

(b) State the ratio test for the convergence of a series of positive terms. Hence or otherwise prove that the series

$$x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

converges for every positive value of x.

4. $t_1, t_2, t_3, \ldots \ldots t_k \ldots$ is an infinite sequence of real numbers. The sum of the first k terms $(k = 1, 2, 3, \ldots)$ is S_k.

   Given that $S_k = \log_{10}(k^2 + k)$

   (i) prove that each term of the sequence is positive,

   (ii) prove that the sequence is decreasing (e.g. $t_k < t_{k-1}$ for $k \geq 2$),

   (iii) find $\lim\limits_{k \to \infty} t_k$.

5. (a) Assuming that $\lim\limits_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$ differentiate $\sin x$ with respect to x from first principles.

   (b) Differentiate with respect to x

   (i) $\dfrac{2}{x\sqrt{x}}$;

   (ii) $e^{-x}\sin 2x$.

   (c) Prove that $\dfrac{d}{dx}(a^x) = a^x \log_e a$.

6. If for a car travelling at a steady speed of v miles per hour the rate k of consumption of petrol in gallons per hour is given by the formula $k = \frac{1}{2}[1 + (0{\cdot}0001)v^2]$ find an expression for the total amount of petrol used in a journey of 150 miles.

   What value of v in m.p.h. would minimise the amount of petrol used? Compute this minimal amount in gallons. (Give each answer to 2 places of decimals.)

7. f and g are functions defined as follows:

   $f(x) = x(1 - x^2)^{-\frac{1}{2}}, \; x \in R,$

   $g(x) = (1 - x^2)^{\frac{1}{2}}, \; x \in R.$

   Show that $f(-x) = -f(x)$,

   Determine the slope of the graph of f where it intersects the x-axis and sketch the graph.

   Show that $g^1(x) = -f(x)$ and compute

   $$\int_0^{\frac{1}{2}} f(x)dx.$$

8. Evaluate

   (i) (a) $\displaystyle\int_\alpha^\beta \frac{3}{\sqrt{t}}\,dt$; (b) $\displaystyle\int_\alpha^\beta \sqrt[3]{t}\,dt$;

   (ii) $\displaystyle\int_0^1 \frac{dx}{\sqrt{2 - x}}$;

(iii) $\int_0^{\frac{\pi}{6}} \cos 2\theta \cos 4\theta \, d\theta.$

9. Find the area of the smaller region enclosed by the parabola $y^2 = 2x$ and the circle $x^2 + y^2 = 3$.

10. A thin vertical pole of height 10 feet is fixed in level ground. Two taut light wires are tied to a peg in the ground at a distance x feet from the foot of the pole. One wire stretches to the top of the pole and the other to a point midway up the pole.

The angle between the wires is θ. If the size of θ is to be a maximum calculate the distance x. (See diagram).