1971

PAPER I

Six questions to be answered.

1. (a) The line $3x - 2y + 6 = 0$ is the image of the line L by the translation $(1, 0) \rightarrow (0, 1)$.
Find the equation of L.

(b) Show that the lines

$$x + y + 4 = 0$$
$$9x - 5y - 20 = 0$$
$$5x - 9y + 20 = 0$$

form an isosceles triangle.
If α and β are the measures of the angles opposite the equal sides, show that $\tan \alpha = 3\frac{1}{2} = \tan \beta$.

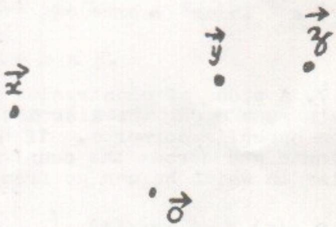
2. K is the circle $x^2 + y^2 + 2x - 6y + 1 = 0$.
Find (i) its radius, (ii) the co-ordinates of its centre, (iii) the length of the y-axis cut off by K.
S is the circle $x^2 + y^2 - x - 3y - 2 = 0$ which contains the

points of intersection of K and a line Q. Find the equation of Q and show that it contains the centre of S.

3. If $t \in R$ show that the point $(at^2, 2at)$ is a point of the parabola $y^2 = 4ax$ and write down the equation of the tangent to the parabola at this point.
A chord $[pq]$ of the parabola $y^2 = 4ax$ intersects the x-axis at s. T_1 and T_2 are tangents to the parabola at p and q, and $T_1 \cap T_2 = \{k\}$. Prove that the distance of s from the focus is equal to the distance of k from the directrix.

4. (a) Prove that $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$.
(b) If $z = 2 - i3$, show z and z^2 on an Argand diagram.
(c) Solve the equation
$$z^2 = \tfrac{1}{2} + i\frac{\sqrt{3}}{2}.$$

5. (a) If the three vectors $\vec{x}, \vec{y}, \vec{z}$ (as in diagram) are collinear, show that the three vectors $\vec{o}, \vec{y} - \vec{x}\ \vec{z} - \vec{x}$ are also collinear ($\vec{o}$ is the null vector).



(b) p, q, r are three points of a circle of centre o, the origin. The points are represented, respectively, by
$$-3\vec{i} - 4\vec{j}, \quad 3\vec{i} - 4\vec{j}, \quad 5\vec{j}.$$
Show that
$$\cos \angle por = \frac{-4}{5},$$
and <u>hence</u> deduce that
$$\angle por = 2\angle pqr.$$

6. (a) Define a linear transformation of the vector-plane π_o.
A line segment $[ab]$ is divided at p in the ratio $x{:}y$. Prove that the image of $[ab]$ by a linear transformation f is divided at $f(p)$ in the same ratio $x{:}y$.
(b) $\{\vec{e}_1, \vec{e}_2\}$ is a basis of the vector-plane π_o, and f is a linear transformation defined by
$$f(\vec{e}_1) = 3\vec{e}_1 - 4\vec{e}_2$$
$$f(\vec{e}_2) = \vec{e}_1 + 2\vec{e}_2$$
Write down the matrix of f and use this matrix to find $f(\vec{x})$ where $\vec{x} = 13\vec{e}_1 - 29\vec{e}_2$.
(c) $\begin{pmatrix} 1 & -4 \\ 3 & 2 \end{pmatrix}$ and $\begin{pmatrix} -2 & 1 \\ 2 & -3 \end{pmatrix}$ are the matrices of the linear transformations f_1 and f_2 respectively.
Find the image of the point (2, 5) under the composite transformation $f_2 \circ f_1$ Is $f_2 \circ f_1 = f_1 \circ f_2$?
Explain your answer.

7. Write down the matrices of the following transformations:
(i) Reflection in the line $R\vec{i}$ [i.e. the x-axis].
(ii) Reflection in the line $R(\vec{i} + \vec{j})$ [i.e. the line $y = x$].

(iii) Rotation of angle θ about the origin.

f is the reflection in the line $R(\vec{i} - \vec{j})$,
g is the reflection in the line $R\vec{i}$,
k is the reflection in the line $R(\vec{i} + \vec{j})$.
Show that

$$k \circ g = g \circ f$$

and hence find the matrix of the reflection in the line $R(\vec{i} - \vec{j})$.

8. Define (i) complex number,
(ii) addition of complex numbers.
(iii) multiplication of complex numbers,
(iv) conjugate of a complex number.

Show that

$$z_1\bar{z}_2 + \bar{z}_1 z_2 = 2Re(z_1\bar{z}_2) \leq 2|z_1||z_2|,$$

where $Re(z)$ means the real part of a complex number z.
If $z_1 = 3 + 4i$ illustrate on an Argand diagram

(i) z_1; (ii) $\frac{5}{z_1}$; (iii) $z_1 - 2$; (iv) $z_1 + i$.

On another Argand diagram illustrate the sets

$A = \{z$, such that $|z| = 2\}$,
$B = \{z$, such that argument of $z = \frac{\pi}{4}\}$.

Illustrate the set $A \cap B$.

9. Draw a rough graph of each of the following functions:
(i) $x \to |\sin x|$, $x \in R$,
(ii) $x \to \sin|x|$, $x \in R$,
(iii) $x \to |\log_e x|$, $x \in R$, $x > 0$,
(iv) $x \to \log_e |x|$, $x \in R$, $x \neq 0$,
(v) $x \to f(x)$ where $f(x) = n$, $n \leq x < n + 1$ and n is an integer.

In each case give the range (image) and say whether or not each function is periodic.

10 A. (i) The operation $*$ is defined on the set of rationals Q by $a * b = \frac{1}{2}(a + b)$.
Is $Q, *$ a group? Give your reason.

(ii) Let $S = \{1, 2, 3, 4\}$ and let $a \in S$, $b \in S$. Define $a * b$ as the remainder when $a \times b$ is divided by 5 (i.e. $3 * 4 = 2$).
Draw up a Cayley table for the operation $*$ and prove that $S, *$ is a commutative group.

OR

10 B. (a) A penny is tossed four times. Write down the sample space for this experiment (i.e. write down all the possible outcomes). Using this sample space, or otherwise, find the probability of getting

(i) a head on each of the first two throws,
(ii) at least two heads,
(iii) a head on the first and third throws.

(b) A pack of 52 playing cards consists of 4 suits - spade and club which are black suits, heart and diamond which are red suits. There are 13 cards in each suit, three of which are picture cards called Jack, Queen and King. What is the number of ways of picking two cards from the pack?
If two cards are picked at random at the same time from the pack, calculate the probability that
(i) each card is from a black suit,
(ii) one card is either a spade or a club, and the other card is a heart,
(iii) the two cards are both Jacks, or both Queens or both Kings.