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LEAVING CERTIFICATE EXAMINATION, 1970

MATHEMATICS (HONOURS) – PAPER II (300 marks)

MONDAY, 15th JUNE – MORNING 9.30 to 12

Six questions to be answered. All questions are of equal value. Mathematical Tables may be obtained from the Superintendent.

1. What is a complex number ?
Define the sum of two complex numbers.
Find the real and the imaginary parts of the number

$$\frac{3 - 4i}{-7 + i}$$

If a and b are complex numbers prove that $|a + b| \leqslant |a| + |b|$.

Prove

$$|a + b + c| \leqslant |a| + |b| + |c|.$$

2. The graph of the function ϕ which is defined for each $x \epsilon$ R as follows:
$\phi(x) = x^4 + bx^3 + cx^2 + dx + e$
is a curve which passes through the origin and has either maximum or minimum values at $x = -2$, $x = 1$ and $x = 3$. Determine b, c, d and e.
Find the equation of the tangent to the curve at $x = -1$.

3. (a) (i) Say whether or not each of these sequences has a limit. Give reasons for your answers and give the value of the limit where such limit exists. (Assume the patterns persist.)

$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \ldots\ldots$

$1, 0, \frac{1}{2}, 0, \frac{1}{4}, 0, \frac{1}{8}, 0, \frac{1}{16}, \ldots\ldots$

(ii) The n^{th} term of a series is $\frac{x^{2n+1}}{(2n+1)!}$ Prove that the series converges.

(b) $A_1, A_2, A_3, \ldots\ldots A_n, \ldots\ldots$ is an infinite sequence of subsets of the set $N_0 = \{1, 2, 3, 4, \ldots\ldots\}$ such that

$A_n = \{n + k \mid n, k \epsilon N_0\}$ [i.e. $A_n = \{n + 1, n + 2, n + 3, \ldots\ldots\}$]

If $B_n = A'_n$ (the complement of A_n) show that

$B_m \subset B_n$ if $m < n$ $(m \epsilon N_0)$

If $C_m = \{A_n \mid A_n \cap B_m \neq \phi\}$ list the elements of C_5.

4. It is given that for $x > o$ and $x \epsilon$ R the function

$$y = f(x) = \log_e x$$

has the properties

(a) $f(1) = 0$

(b) $\frac{dy}{dx} = f'(x) = \frac{1}{x}$

and it is also given that for any two functions $g(x)$ and $h(x)$, both defined over a closed interval

$$g'(x) - h'(x) = 0$$

implies that $g(x) - h(x)$ is a constant.

Prove that:

(i) $\frac{d}{dx} \log_e (ax) = \frac{1}{x}$ if $a > o$ is a constant and deduce that

$\log_e (ax) = \log_e x + \log_e a$

(Hint: Put $ax = u$)

(ii) $\frac{d}{dx}(\log_e x^k) = \frac{k}{x}$; k any rational number and deduce that

$\log_e (x^k) = k \log_e x$

(iii) $\log_e \frac{x}{a} = \log_e x - \log_e a$

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5. If $x \in R$, but $x \neq 1$ differentiate from first principles with respect to x

$$\frac{x}{1+x}$$

Differentiate with respect to x

(i) e^{2x}; $x \in R$

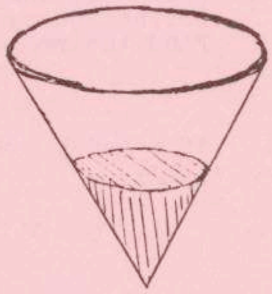
(ii) $x \log ax$; $x > 0$, $a > 0$ a constant

(iii) $e^{-2x} \cos x$; $x \in R$

Find the values of c for which e^{-cx} is a solution of

$$\frac{d^2y}{dx^2} - 8\frac{dy}{dx} - 9y = 0.$$

6. A container is in the shape of a huge inverted cone of height 5 feet and radius of base 4 feet. If water is flowing out of the container at a constant rate of 3 cu. ft. per minute find the rate at which the level of the water is falling when the depth is 2 feet.



7. (a) Evaluate

(i) $\int_{-k}^{k} (a + bx + cx^2)dx$; a, b, c constants.

(ii) $\int_{1}^{2} \pi x^2 \, dy$ where $xy = c^2$; c constant.

(iii) $\int_{-c}^{+c} \frac{c}{2}(e^{\frac{x}{c}} - e^{-\frac{x}{c}})dx$

(b) If f and g are functions such that $f(x) < g(x)$ for each x belonging to the closed interval $[a, b]$ show by illustrations that the following are true

(i) $\int_{a}^{b} f(x)dx \geq 0$ if $f(x) \geq 0$.

(ii) $\int_{a}^{b} f(x)dx < \int_{a}^{b} g(x)dx$.

Hence deduce

$$\int_{0}^{\frac{1}{2}} \frac{x\,dx}{1+x^2} < \int_{0}^{\frac{1}{2}} \frac{dx}{1+x^2}$$

8. £400 and 500 man-hours are available to sow a 10-acre field which is suitable for growing one or both of two crops A and B. Crop A requires 80 man-hours per acre to grow and costs £60 per acre to sow. Crop B requires 30 man-hours per acre to grow and costs £40 per acre to sow. Find the greatest amount of crop B which can be sown if the profit is to be a maximum and if the profit per acre from crop A is double that from crop B.

(Note: x men working for y hours is xy man-hours)

9. (a) Three dice are tossed together. Calculate the probability that the sum of the scores will be less than seven.

(b) If on average rain falls on 10 days in every 30 find the probability

(i) that a week will be dry,
(ii) that rain will fall on just 2 days of a week.

10. f and ϕ are two functions defined as follows

$$f(x) = x^2 + 1, \ x \in R; \quad \phi(x) = -x, \ x \in R.$$

(a) Determine the range of each function.

(b) Find (i) $\phi(2)$, (ii) $3f(-2)$, (iii) $f\left(\phi(2)\right)$, (iv) $\phi\left(f(2)\right)$

(c) Is $f\left(\phi(2)\right) = \phi\left(f(2)\right)$?

(d) Write $\phi\left(f(x)\right)$ as an expression in x.

Can the equation $f(x) = \phi(x)$ be solved for $x \in R$? Explain your answer, and illustrate it by a diagram.