AN ROINN OIDEACHAIS

LEAVING CERTIFICATE EXAMINATION, 1970

MATHEMATICS (HONOURS) – PAPER I – (300 marks)

WEDNESDAY, 10th JUNE – Morning, 9.45 to 12.15

Six questions to be answered.
All questions carry equal marks.
Mathematical Tables may be had from the Superintendent.
R is the set of real numbers.

1. A vessel in the shape of an inverted right circular cone of semi-vertical angle 30° contains water to a depth of 6 cm. A solid sphere is lowered into the vessel until it rests against the side of the vessel and the water then just rises to where the sphere touches the side. If the level of water has risen a vertical height of h cm., express, in terms of h, the volume of the submerged section of the sphere.

2. P and Q are conjugate points with respect to a circle of centre O and of radius r and P is outside the circle. Illustrate the polar of P and the polar of Q with respect to the circle.

PB is any line from P cutting the circle at A and B. Tangents at A and B meet in S. Prove that S is on the polar of P with respect to the circle.

Prove any theorem on Pole and Polar that you use.

3. H is a point in a line segment AB and K is a point in AB produced such that

$$AH : HB = AK : KB.$$

If M is the midpoint of AB, prove that

$$MH \cdot MK = MA^2.$$

State a converse of the theorem.

When are two circles said to cut orthogonally ?

J is a circle which cuts a circle K orthogonally. A diameter of K meets J at P and Q. Prove that this diameter is divided harmonically at P and Q.

4. (a) (x, y) are the coordinates of any point on the locus defined by

$$x = 3t - 1,\ t \in R$$
$$y = 1 - 2t,\ t \in R.$$

Verify that $(-1, 1)$ are the coordinates of a point of the locus. Write down the coordinates of any other two points of the locus and show that they are collinear with $(-1, 1)$.

(b) What locus is defined by each of the following equations:

(i) $(x + 1)^2 - y = 0$

(ii) $(x + 1)^2 - y^2 = 0$

(iii) $(x + 1)^2 - y^2 = 1$?

In (iii) find the domain of x for which there is no real y. Sketch the locus in each case.

5. The line L = 0 cuts the circle S = 0 at the points A and B. Show that for every $\lambda \in R$, the equation $S + \lambda L = 0$ represents a circle through A and B.

Find the equations of the circles which have the x-axis as tangent and which pass through the points of intersection of the line $x + y - 6 = 0$ and the circle $x^2 + y^2 - 8x - 8y + 28 = 0$.

6. If (x_1, y_1) and (x_2, y_2) are two points on the parabola $y^2 = 4ax$, prove that

$$y_1 + y_2 = \frac{4a(x_1 - x_2)}{y_1 - y_2}.$$

If $L_1 = 0$ and $L_2 = 0$ are the equations of two lines, how do you interpret the equation $L_1 + \lambda L_2 = 0$, where $\lambda \in R$?

A and B are two points on the parabola $y^2 = 4ax$. Tangents at A and B meet at T. Prove that the line through T, parallel to the x-axis, bisects the chord AB.

OVER →

$(0, 0), (0, 1), (1, 0), (-1, -1)$.

(ii) Plot the points whose polar coordinates are:

$$(3, \tfrac{\pi}{2}),\ (3, -\tfrac{3\pi}{2}),\ (2, \tfrac{5\pi}{4}).$$

(iii) Express the polar equation

$$r^2 = a^2 \cos 2\theta$$

in cartesian form (i.e. in x and y).

(iv) Prove that the curve

$$r = \frac{2}{2 + \cos\theta}.$$

is symmetrical about the axis $\theta = 0$ and sketch the curve.

8\. (i) Distinguish between the vector $\vec{a}$ and its magnitude $|\vec{a}|$.

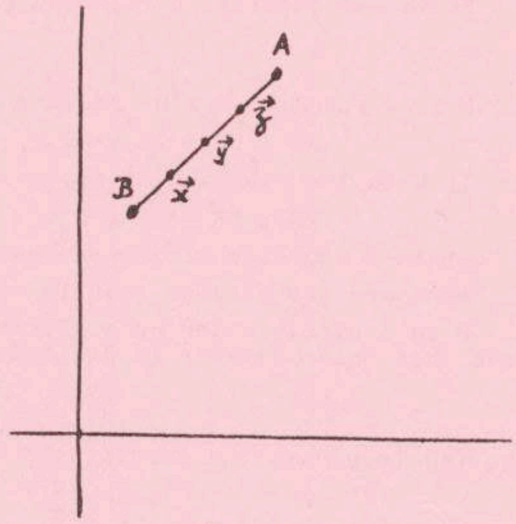
Define the scalar product (dot product) $\vec{a} \cdot \vec{b}$ of two vectors $\vec{a}$ and $\vec{b}$ and deduce that $\vec{a} \cdot \vec{a} = |\vec{a}|^2$.

If $\vec{a}, \vec{b}, \vec{c}$ are vectors such that $\vec{c} = \vec{a} + \vec{b}$, find $|\vec{c}|^2$ and hence deduce that in a right angled triangle the square on the hypotenuse is equal to the sum of the squares on the other two sides.

(ii) The three vertices A, B, C of a triangle are, respectively, $4i + 8j$, $i + 5j$, $5i + j$, where i and j have their usual meanings.

Express the vectors $\vec{CB}$, $\vec{BA}$ and $\vec{AC}$ in terms of i and j.

The side BA of the triangle is subdivided into 4 segments of equal length at $\vec{x}, \vec{y}, \vec{z}$, as in diagram. Express $\vec{x}, \vec{y}, \vec{z}$ in terms of i and j.



9\. (a) Find the general solution of

$$\sin\theta(\sin 2\theta - \cos 2\theta) = 0.$$

(b) If $k = \cos\theta + i \sin\theta$, where $i = \sqrt{-1}$, prove that

$$\frac{1}{k} = \cos\theta - i \sin\theta$$

and deduce that

$$k^n + \frac{1}{k^n} = 2 \cos n\theta.$$

Express $\sin n\theta$ in terms of k and show that

$$\cos^5\theta = \tfrac{1}{16}(\cos 5\theta + 5 \cos 3\theta + 10 \cos\theta).$$

10\. $\{x_1, x_2, x_3, \ldots, x_n\}$ is a set of n real numbers. Show how you would calculate

(i) $\bar{x}$, the mean of the set

(ii) σ, the standard deviation.

If the numbers $x_1, x_2, x_3, \ldots, x_n$ have respective frequencies $f_1, f_2, f_3, \ldots, f_n$, show that

$$\sigma^2 = \frac{\Sigma f_r (x_r - \bar{x})^2}{N}, \text{ where } N = \Sigma f_r,$$

and deduce that this can be written in the form

$$\sigma^2 = \frac{\Sigma f_r x_r^2}{N} - \left(\frac{\Sigma f_r x_r}{N}\right)^2.$$

A gardener wishing to cultivate a yellow rose sowed sets of 5 seeds 500 times. The frequency of yellow roses per set is given in the following table:

Number of yellow roses (x_r)	0	1	2	3	4	5
Frequency (f_r)	132	167	140	45	12	4

Calculate the standard deviation of the distribution correct to two places of decimals.