MATHEMATICS (HONOURS) – PAPER II (300 marks)

Six questions to be answered. All questions are of equal value. Mathematical Tables may be obtained from the Superintendent.

N is the set of natural numbers. R is the set of real numbers. C is the set of complex numbers.

1. (a) For each complex number $z = x + iy\ (x,\ y \in \mathrm{R})$ let $\overline{z} = x - iy$.
   (i) Show that $z + \overline{z}$ and $z\overline{z}$ are real numbers,
   (ii) Show that $\overline{z_1\ z_2} = \overline{z}_1\ \overline{z}_2\ (z_1,\ z_2 \in \mathrm{C})$,
   (iii) If $z = \frac{z_1}{z_2}$, show that $\overline{z} = \frac{\overline{z}_1}{\overline{z}_2}$.

2. Find the three roots of the equation $x^3 + 2x^2 - 8x - 13 = 0$ correct to one place of decimals in each case.

3. (a) Show that the sequence $\frac{1+1}{1^2+1},\ \frac{2+1}{2^2+1},\ \frac{3+1}{3^2+1}\ \ldots$
   where the same pattern obtains in subsequent terms, is convergent.
   (b) Prove that $\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$ is convergent.
   Show that $\frac{1}{n^2} \leqslant \frac{4}{n(n+2)}$ for all $n \geqslant 1$ and hence deduce that $\sum_{n=1}^{\infty} \frac{1}{n^s}$ converges for $s \geqslant 2$.

4. (a) Show that the set $\mathrm{A} = \{1, -1, i, -i\}$, where $i = \sqrt{-1}$, is closed under multiplication.
   Is the set $\{1, -1, i\}$ closed under multiplication ?
   If $x,\ y \in \mathrm{A}$, y is called the inverse of x if $xy = 1$. What are the inverses of $1, i$ and $-i$?
   How many subsets, each containing three elements, can be formed from A ? Take any one of these subsets, find the number of its permutations taking the three elements each time and hence test this set of three elements for associativity under multiplication.
   (b) Let $\mathrm{P} = \{1, a, 4\}$ and $\mathrm{S} = \{1, a\}$. If $\mathrm{P} \cap \mathrm{X} = \mathrm{S}$, find X. Is X unique ? Give your reason.
   Is there a set Y such that $\mathrm{P} \cup \mathrm{Y} = \mathrm{S}$? Give your reason.

5. (a) Differentiate from first principles $\frac{1}{1 - x^2}$ with respect to x.
   Differentiate with respect to x:
   $(1 - x^2)(1 - x - x^2)$; $\sin\sqrt{4x^2 - 1}$; $\log_e \frac{1 - x^2}{1 + x^2}$.
   (b) Given that $\frac{d^2s}{dt^2} = 16\sin 4t$, find $\frac{ds}{dt}$ and express s as a function of t.
   If $s = 0$ and $\frac{ds}{dt} = 0$ when $t = 0$, show that $s = 4t - \sin 4t$.

6. The height of a right circular cone is 15 cm..
   Prove that the height of the cylinder of maximum volume that can be inscribed in the cone is 5 cm.

7. Evaluate:
   (i) $\int_1^2 x(1 + x)^2 dx$; (ii) $\int_0^{\frac{\pi}{2}} \sin 2x\ \cos 3x\ dx$; (iii) $\int_0^{\frac{\pi}{2}} \frac{\sin 2x\ dx}{1 + \sin^2 x}$
   (iv) $\int_0^1 x e^{3x^2 - 1}\ dx$ (v) $\int_0^1 \frac{(x - 1)dx}{\sqrt{1 + 6x - 3x^2}}$.

8. A hospital wishes to experiment with a mixture of two types of food so that the vitamin content of the mixture consists of at least
   6 units of Vitamin A, 8 units of Vitamin B, $4\frac{1}{2}$ units of Vitamin C.
   The vitamin content in units of each type of food is given in the table:

	A	B	C
Type 1	3	2	1
Type 2	2	4	3

   If type 1 food costs 55 pence per kilogramme and type 2 food costs 45 pence per kilogramme, find the minimum cost of the mixture in pence.

9. Sketch the curves $y = x^2(x - 2)$ and $y = x^3(x - 2)$, $x \in \mathrm{R}$, between $x = -1$ and $x = 2\frac{1}{2}$ paying special attention to the parts of the curves near the origin, to the maximum and minimum points and the points of inflexion.

10. (a) All the natural numbers between 1 and 20 (including 1 and 20) are written on slips of paper one number on each slip. The slips are thoroughly mixed in a drum and one slip is drawn at random. What is the probability that the number on the slip is either prime or a multiple of 3 ?
   (b) A plane can fly if half its number of engines are working. If q is the probability that an engine fails, show that $1 - q^2$ is the probability that a two engine plane makes a successful flight.
   Show also that a two engine plane is safer than a four engine plane if $\frac{1}{3} < q < 1$.