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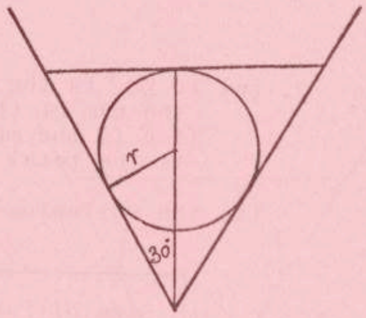
LEAVING CERTIFICATE EXAMINATION, 1967

MATHEMATICS (HONOURS) - PAPER I - (300 marks)

FRIDAY, 9th JUNE - Morning, 10 to 12.30

Six questions to be answered.
All questions carry equal marks.
Mathematical Tables may be had from the Superintendent.



1. A vessel in the shape of an inverted right circular cone of vertical angle 60° contains water to a level of 3" above the vertex. Calculate the volume of the water.

When a solid sphere of radius r inches is then put into the vessel the water rises so as just to cover the sphere, which touches the sides of the vessel (see diagram). Find r correct to two significant figures.

2. (i) State and prove either Ceva's theorem or Menelaus' theorem.

(ii) Prove that the radical axis of two circles which are not concentric is a straight line perpendicular to the line of centres.

Does a pair of concentric circles have a radical axis? Give the reason for your answer.

3. If A, B, C, D are four collinear points the cross-ratio {ABCD}, or {AC, BD}, is defined to be $\frac{AB.\ CD}{AD.\ CB}$. Starting from this definition, without assuming any theorem on cross-ratio, prove the following:- if a transversal cuts a pencil of four rays of vertex O in the points A, B, C, D in that order and a line through B parallel to OD cuts OA in P and OC produced in Q, then $\{ABCD\} = -\frac{PB}{BQ}$.

If $\{ABCD\} = -2$, show that PB = 2BQ. If another ray through O cuts CD in E and PQ produced in K, such that $\{ABED\} = -\frac{1}{3}$, show that BK = 6BQ. Find the value of {PBQK} and hence the value of {ABCE}.

4. The co-ordinates of the vertices of a triangle are (-4, 3), (0, -5), (3, 4). Find the co-ordinates of H, the orthocentre.

Show that O, the origin, is the circumcentre of the triangle. If M is the centroid (the point of intersection of the medians), find the ratio in which M divides OH.

5. (a) Prove that the point (0, 1) lies outside the circle $x^2 + y^2 - 4x + 6y - 3 = 0$, and find the equations of the tangents to the circle from that point.

(b) Say what type of curve (parabola, ellipse, etc., or straight lines) forms the graph of each of the following equations (no proof required):

(i) $(x - 3y)^2 - 5(x - 3y) + 6 = 0$,

(ii) $(x - 3y)^2 + x + 7y + 8 = 0$,

(iii) $\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$.

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6. (i) What is the polar equation of the circle $x^2 + y^2 = a^2$ with the origin as pole? Show that the polar equation of the tangent to the circle at the point whose polar co-ordinates are (a, α) is $r\cos(\theta - \alpha) = a$.

(ii) The straight line through two points A, B, of co-ordinates (x_1, y_1) and (x_2, y_2), cuts the x-axis in the point (k, o). If O is the origin, show that the area of the triangle OAB is given by $\frac{1}{2}k(y_1 - y_2)$, assuming k>o and $y_1 > y_2$.

PQ is a focal chord of a parabola (i.e. the chord PQ passes through F, the focus). M, N are the feet of the perpendiculars from P, Q, respectively, to the directrix. If A_1 is the area of the triangle OPQ, where O is the vertex, and A_2 the area of the trapezium PQNM, prove that $A_1 : A_2$ = length of OF : length of PQ.

7. (a) If Σd^+ is the sum of the positive deviations from the arithmetic mean, and Σd^- the sum of the negative deviations, show that $\Sigma d^- = -\Sigma d^+$. If δ is the mean deviation from the mean, deduce that $\delta = 2\Sigma d^+/N$, where $N = \Sigma f_r$ is the total frequency.

(b) The following table shows the number of farms of various sizes in a certain area:

Size of Farm (in acres)*	0–10	10–20	20–30	30–50	50–100
Number of Farms	50	120	250	200	180

Draw a histogram to represent the distribution.
Calculate either the mean deviation or the standard deviation (on the assumption that the values are concentrated at the mid-points of the class-intervals).

*Note: '0–10' means less than 10; '10–20' means at least 10 but less than 20, etc.

8. (a) The position-vectors of three points a, b, c, are $i + 5j$, $-3i + 2j$, $-i - 2j$, respectively, where i and j are unit vectors at right angles to each other. Express the vectors $\vec{ba}$ and $\vec{bc}$ in terms of i and j.

Find (i) the value of $\cos a\hat{b}c$

(ii) the position-vector of the point d such that $abcd$ is a parallelogram.

(b) What is meant by saying "addition of vectors is associative"? Draw a diagram to illustrate this property and explain clearly how your diagram illustrates it.

9. What is the range of values of each of the following, the domain of x being the real numbers:

(i) $\cos x$, (ii) $\sin 2x$, (iii) $\sin x . \cos x$, (iv) $\sin x \cos \alpha + \cos x \sin \alpha$, where α is a constant?

Find the range of $a \sin x + b \cos x$ in terms of the constants a and b.
What is meant by saying that '$\sin x$ has a period of 2π'?
Write down two functions of x of the form $a \sin nx + b \cos nx$, where a, b, n are non-zero integers, as follows:

(i) a function of period 2π and range $[-5, 5]$,

(ii) a function of period π and range $[-5, 5]$.

10. (a) Find the general solution of the equation $\cos x - \cos 3x - \cos 5x = \sin 4x - \sin 8x$.

(b) The sides of a triangle are 2", 3", 4", in length, respectively. Find the size of the largest angle of the triangle, correct to the nearest degree.

(c) Prove De Moivre's Theorem where the exponent is a positive integer.

Hence, or otherwise, show that $(\sqrt{3} + i)^{12} = 2^{12}$.