Irish Mathematics Teachers Association
Cumann Oidí Matamaitice na h-Éireann

NOTES ON THE LEAVING CERTIFICATE MATHEMATICS COURSES

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NOTES

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LEAVING CERTIFICATE MATHEMATICS COURSES

CONTENTS

INTRODUCTION

These Notes have been produced for two main reasons:

- to provide teachers of Leaving Certificate Mathematics courses with some background information which may help them in teaching less familiar areas of the courses;
- to suggest mathematical approaches which may be more direct, or more complete, or more accessible than some of the approaches customarily used in schools.

The Notes were originally prepared by the NCCA Course Committee while it was developing the courses. Among the design principles that the Committee used — principles which were set out in the booklet *The Leaving Certificate: Mathematics Syllabus* (pp. 3-4) and also in the Draft *Guidelines for Teachers* for the Foundation Course (pp. 9-10) — were requirements that the courses should be teachable and learnable, and that the mathematics should be sound.

- For the courses to be *teachable and learnable,* they have to be short enough to be addressed in the time available; so the Committee looked for quick routes to important results. The proof of the Factor Theorem without using the Remainder Theorem is an instance.
- Also, the work must be presented at a level appropriate to the students; so difficult topics need to be addressed, where possible, in a comparatively simple way. An example in this case is the systematic method offered for finding the solution sets for inequalities.
- Moreover, where material is unfamiliar to teachers, a more thorough presentation may be helpful. This has been provided most notably for transformation geometry (which occupies a major part of the booklet), but also for difference equations and probability.
- As regards the *soundness of the mathematics,* it was observed that the traditional presentation of some topics is slightly incomplete; in such cases teachers — or able students — might appreciate a fuller treatment. The derivative of $\sqrt{x}$ is an example.

Thus, the emphasis is on the *mathematics* of various parts of the courses. It is *not* the main purpose of the Notes to indicate exactly how the various topics might be taught. However, in some cases, suggestions about teaching methods are also included. Fuller considerations of methodology for the Higher and Ordinary courses should appear in the forthcoming *Guidelines* *for Teachers* (an accompanying volume to the draft Guidelines already produced for the Foundation course, and intended to promulgate some of the good practice that has developed in the years in which the courses have been running).

The Notes have had a limited circulation for some time; but the representatives of the Irish Mathematics Teachers' Association on the Course Committee felt that they could be made more widely available as a resource, so the Association and the NCCA have cooperated in producing this booklet. It is the second publication issued jointly by the IMTA and the NCCA; it follows the *Specimen Questions* brought out in Spring 1994. The IMTA hopes that the booklet will be of use to its members and to mathematics teachers throughout the country. The Association looks forward to further collaboration with the NCCA and other bodies at it seeks to support teachers of mathematics working to provide the best possible mathematical education for the students in our schools.

Acknowledgments
Special thanks are due to two members of the Course Committe for their work on these Notes. Most of the material was devised and drafted by Professor Paddy Barry; the form in which it is presented is due largely to John Evans, who not only contributed to the writing process but also prepared the text and graphics for the printers. Their inspiration, and their commitment to the production of the Notes, are gratefully acknowledged.

Autumn 1995

PART A

HIGHER LEVEL

GEOMETRY

CORE TRANSFORMATION GEOMETRY

The students should be familiar with the following elements of co-ordinate geometry, both from their Junior Cert. course and the material which makes up the core Line section of the Leaving Certificate course.

Distance between two points.
Midpoint of a line segment.
Slope of a line through two points.

Area of a triangle with one vertex at the origin.
Area of any triangle given the co-ordinates of the vertices.
Area of a parallelogram (as the sum of the areas of two triangles or otherwise).
Area of a square.
Area of a rectangle.

Equation of a straight line:

$$y = (\text{slope})x + c$$

$$y - y_1 = (\text{slope})(x - x_1)$$

$$ax + by + c = 0 \qquad \text{where the slope} = -\frac{a}{b}$$

Test for a point being on a line.
Tests for lines being parallel or perpendicular.

Perpendicular distance of a point from a line - use of formula - significance of sign.

(Matrices are not compulsory for this section, but they can be useful).

The point (x,y) as the vector $\begin{pmatrix} x \\ y \end{pmatrix}$.

The transformation $\mathbf{f}: (x,y) \rightarrow (x',y')$ represented by the equations:

$$x' = ax + by$$

$$y' = cx + dy$$

as the transformation $\mathbf{f}: \begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} x' \\ y' \end{pmatrix}$

represented by the matrix equation

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Calculation of the inverse of a matrix.

Parametric equations of a line and line segment (dealt with below).

Introduction of the topic

The introduction of this topic may provide a useful opportunity to revise co-ordinate geometry - especially the concepts listed above. In addition, basic skills in matrices can be reviewed as well.

Transformations

The transformations we will deal with act on points of the plane.

If we want a transformation **f** to act on some point p, whose co-ordinates are (3,2) , we have to describe what **f** will do to the x co-ordinate (3) **and** to the y co-ordinate (2).

In general we will describe a transformation **f** as acting on a point p(x,y) and sending p to an image point called p' (or **f**(p)) where the coordinates of p' are (x',y').

How do we know what these new coordinates are?

Example:

The transformation **f** maps (x,y) to (x',y') such that

$$x' = 3x + 2y$$

$$y' = x - y$$

Find the image of the points p(5,-2) , q(-1,-4) and o(0,0).

Solution (1):

The calculations may be done directly. Take p(5,-2) for example.
The x co-ord is 5 and the y co-ord is -2. Thus x=5 and y=-2.

Applying the equations we obtain x' = 3(5)+2(-2) = 11

and y' = (5)-(-2) = 7.

So p(5,-2) is mapped under **f** to p'(11,7)

or

p(5,-2) is mapped under **f** to **f**(p), whose co-ordinates are (11,7).

We can now repeat this process to the points q and o.

Solution(2):

We represent the transformation **f** as a matrix acting on a vector to produce an image vector.

The matrix is easily obtained.

The transformation equations given to us were: x' = **3**x + **2**y
y' = **1**x **-1**y

so **f** may be represented as

$$\begin{pmatrix} 3 & 2 \\ 1 & -1 \end{pmatrix} \text{ acting on the vector } \begin{pmatrix} x \\ y \end{pmatrix}.$$

If we wish to obtain the image of p(5,-2) we treat p as a vector and multiply by the transformation matrix:

$$\begin{pmatrix} 3 & 2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 5 \\ -2 \end{pmatrix} = \begin{pmatrix} 11 \\ 7 \end{pmatrix}$$

A particularly useful property of the matrix approach is that the images of several points can be calculated very quickly.

The three points given to us above were p(5,-2) , q(-1,-4) and o(0,0). We may deal with these all at once by forming a 2x3 matrix

$$\begin{pmatrix} 5 & -1 & 0 \\ -2 & -4 & 0 \end{pmatrix}$$

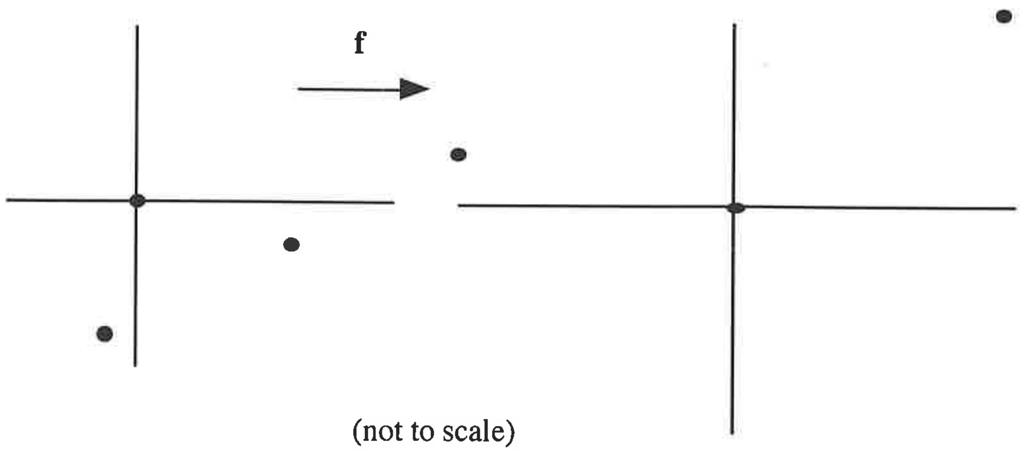
and multiply this by the transformation matrix

$$\begin{pmatrix} 3 & 2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 5 & -1 & 0 \\ -2 & -4 & 0 \end{pmatrix} = \begin{pmatrix} 11 & -11 & 0 \\ 7 & 3 & 0 \end{pmatrix}$$

giving the image vectors

$$p' = \begin{pmatrix} 11 \\ 7 \end{pmatrix} \qquad q' = \begin{pmatrix} -11 \\ 3 \end{pmatrix} \qquad r' = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

or, as points, p'(11,7) , q'(-11,3) , r'(0,0).



(not to scale)

Now what is it we have to do in the Core?

<u>What is in the Core Transformation Geometry?</u>

*Each transformation **f** of the plane Π which has the coordinate form

$$(x,y) \to (x',y') \quad \text{where}$$

$$x' = ax + by$$
$$y' = cx + dy$$

$$\text{and} \quad ad - bc \neq 0 \ ,$$

maps each line to a line, each line segment to a line segment, each pair of parallel lines to a pair of parallel lines, and consequently each parallelogram to a parallelogram.

Proof confined to a specific transformation (numerical values for a, b, c, and d) .

Examples of the invariance or non-invariance of perpendicularity, distance, ratio of two distances, area, and ratio of two areas connected with specific parallelograms (including rectangles and squares) under transformations of the form

$$x' = ax + by$$
$$y' = cx + dy$$

with numerical coefficients.

We propose to deal with some of the **last** paragraph of material first, as students may find this quite easy to do, and be better prepared for the quite subtle ideas needed to understand the proof section set out in the **first** paragraph.

It should be noted, however, that a certain care should be taken with the language we use at this stage. We will talk about a triangle whose vertices are the points p, q and r and the triangle whose vertices are the image points p', q', and r' under some transformation. We may then calculate the areas of these triangles and so on, but we are not saying that the image of the triangle pqr **is** the triangle p'q'r' - the answer to that depends on the nature of the transformation and what we understand a triangle to be - is it the set of points that make up the sides? or these points plus the set of points in the interior? (All of these issues are addressed in the OPTION on Further Transformational Geometry).

Compendium Question: (NB: equations of lines/line segments not used here)

Consider the points o(0,0) , p(3,4) , q(-2,6) , r(-5,2) , s(4,-3) and the transformation **f**

where **f** : (x,y) $\rightarrow$ (x ′,y ′) and

where $x' = 3x - y$

$y' = x + 2y$

Calculate the following and make a comment about invariance/non-invariance under **f**:

(i) | pq | and | p'q' | { **or** | **f**(p)**f**(q) | }

Solution:

After reading through the question we see that we need the images of o,p,q,r,and s.

Our matrix for the transformation is $\begin{pmatrix} 3 & -1 \\ 1 & 2 \end{pmatrix}$

$$\text{so}: \quad \begin{pmatrix} 3 & -1 \\ 1 & 2 \end{pmatrix} \overset{\begin{matrix} o & p & q & r & s \end{matrix}}{\begin{pmatrix} 0 & 3 & -2 & -5 & 4 \\ 0 & 4 & 6 & 2 & -3 \end{pmatrix}} = \overset{\begin{matrix} o' & p' & q' & r' & s' \end{matrix}}{\begin{pmatrix} 0 & 5 & -12 & -17 & 15 \\ 0 & 11 & 10 & -1 & -2 \end{pmatrix}}$$

| pq | = √29 and | p'q' | = √290 . As these are not equal then distance between points is **not invariant** under **f**

(Remember that only one counter-example is required to disprove an hypothesis !)

(ii) Calculate the following and make a comment about invariance/non-invariance under **f**:

$$\text{the ratio } \frac{|pq|}{|qr|} \text{ and the ratio } \frac{|p'q'|}{|q'r'|} \left\{ \text{or } \frac{|\mathbf{f}(p)\mathbf{f}(q)|}{|\mathbf{f}(q)\mathbf{f}(r)|} \right\}$$

Solution:

| pq | = √29 and | qr | = 5 and | p'q' | = √290 and | q'r' | = √146.

The ratio of | pq | to | qr | is 1.077033.... (EC)

The ratio of | p'q' | to | q'r' | is 1.4093621....(EC).

As these ratios are not equal, then the ratio of distances between points is not invariant under **f**.

(iii) Calculate the following and make a comment about invariance/non-invariance under **f**:

the ratio $\frac{|op|}{|rq|}$ and the ratio $\frac{|o'p'|}{|r'q'|}$ $\left\{ \text{or } \frac{|\mathbf{f}(o)\mathbf{f}(p)|}{|\mathbf{f}(r)\mathbf{f}(q)|} \right\}$

Solution:

| op | = 5 and | rq | = 5 so their ratio is 1.

| o'p' | = $\sqrt{146}$ and | r'q' | = $\sqrt{146}$ so their ratio is also 1.

This result does not establish invariance. Proof of an hypothesis needs a theorem.
Perhaps we have been lucky with our choice of points. Perhaps there is something special about these points .

(There is: they lie on parallel lines: see OPTION).

(iv) Calculate the following and make a comment about invariance/non-invariance under **f**:

the area of the triangle whose vertices are are o,p, and r; and the area of the triangle whose vertices are o' , p' and r' {**or f**(o), **f**(p) and **f**(r) }.

Solution:

Δ with vertices o,p and r has area 13 units^2.
Δ with vertices o', p' and r' has area 91 units^2.

These areas are not invariant under **f**.

(And not just by accident either : the determinant of the matrix of transformation **f** is 7, and it can be shown (see OPTION) that this represents the effect of the transformation as a multiplier of the area of plane figures: here for example 13 x 7 = 91).

(v) Calculate and make a comment about invariance/non-invariance under the following ratios:

$$\frac{\text{the area of the } \Delta \text{ with vertices o,p and q}}{\text{the area of the } \Delta \text{ with vertices o,p and r}} \quad \text{and}$$

$$\frac{\text{the area of the } \Delta \text{ with vertices o}',\text{p}' \text{ and q}'}{\text{the area of the } \Delta \text{ with vertices o}',\text{p}' \text{ and r}'}$$

Solution:

Δ with vertices o, p and q has area = 13 units^2.
Δ with vertices o, p and r has area = 13 units^2.
Ratio is 1.

Δ with vertices o', p' and q' has area = 91 units^2.
Δ with vertices o', p' and r' has area = 91 units^2.
Ratio is 1.

Again the fact that these are the same is not a proof that ratios of areas of plane figures are invariant.

(They are : for proof see OPTION).

(vi) Calculate the following and make a comment about invariance/non-invariance under **f**.

the area of the figure with vertices o,p,q and r and
the area of the figure with vertices o', p', q' and r' .

Solution:

the areas are 26 units^2 and 182 units^2 respectively. So the area of such figures is not invariant under **f**. (Again note : 7 x 26 = 182).

(vii) Calculate the following and make a comment about invariance/non-invariance under **f**.

Investigate if the line through o and p is perpendicular to the line through o and s. Is the line through o' and p' perpendicular to the line through o' and s' ?

Solution:

slope op is 4/3 , slope os = -3/4 , so they are perpendicular.

slope o'p' = 11/5 , slope o's' = -2/15 , not perpendicular.

So being perpendicular is not an invariant property under **f**.

(viii) Calculate the following and make a comment about invariance/non-invariance under **f**.

Investigate if the line through o and p is parallel to the line through r and q. Is the line through o' and p' parallel to the line through r' and q' ?

Solution:

slope op is 4/3 and slope rq is 4/3, hence parallel .

slope o'p' is 11/5, slope r'q' is 11/5, hence parallel.

Again not a proof that pairs of parallel lines are mapped to pairs of parallel lines under **f** (proved below as part of the core for a specific transformation **f**).

The general case is proven in the OPTION.

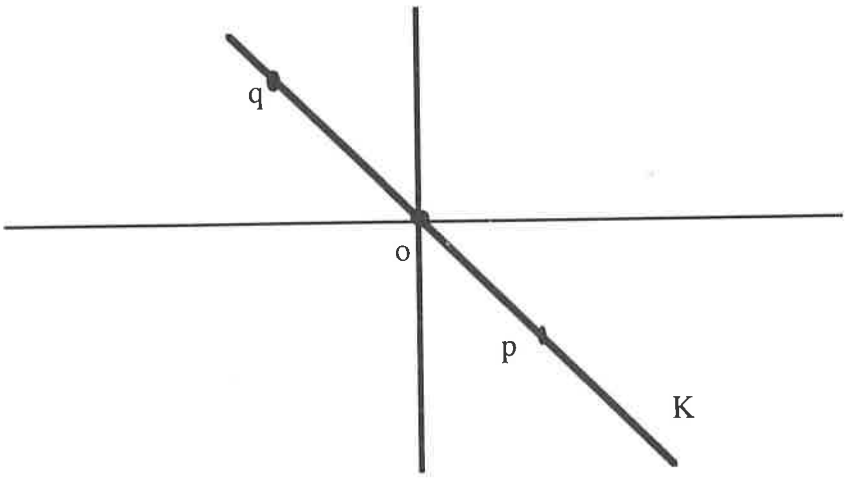
The Theorems.

We have arranged the proofs as four theorems which correspond to the entry in the first paragraph of the syllabus entry.

We need a few bits and pieces before we start, not least being some kind of motivation as to why we need to prove that a line is mapped to a line at all!

A Pathological Case

Consider the line K: $x + y = 0$. Take three points on K: o(0,0) , p(2,-2) and q(-3,3). Let **f** be a transformation of the plane.



We wish to make a distinction between the image of the line K under **f** (which could be anything) and the line that runs through or contains the points **f**(o), **f**(p) and **f**(q), providing these three points are distinct and collinear.

Part of the trouble in seeing a difference between these is that as o, p and q belong to K, then the points **f**(o), **f**(p) and **f**(q) must belong to **f**(K).

However, this does not make **f**(K) a line, even if the points **f**(o), **f**(p) and **f**(q) are distinct and collinear.

Let **f** be the transformation that maps (x,y) to (x', y') where

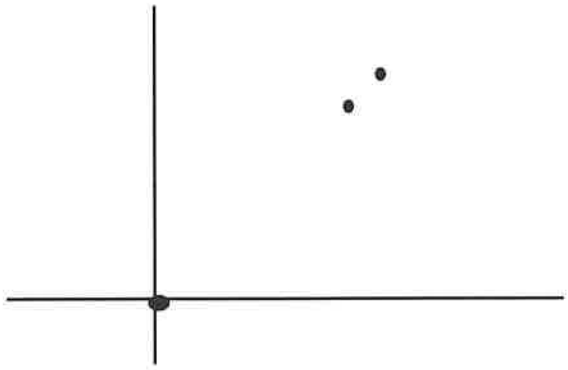
$$x' = \frac{10x^2}{x^2+1} \quad , \quad y' = \frac{10y^2}{y^2+1}$$

(this is **not** a standard transformation)

Applying **f** to o gives **f**(o) = (0,0)

p gives **f**(p) = (8.8)

q gives **f**(q) = (9,9)

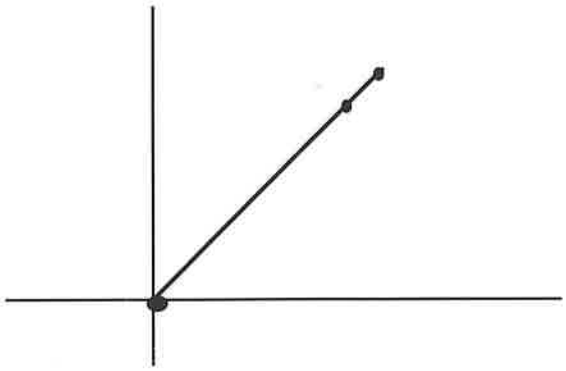


We can see at once that the image points are distinct and collinear.

If we draw a line through these image points we obtain the line x - y = 0.

So is **f**(K) equal to the line x - y = 0 ? The answer is **No**.

We can see this in a variety of ways. If more points are mapped we will see that their images are trapped between (0,0) and (10,10).



By examining the transformation equations we may see that x' and y' must always be positive, so that all the points of x - y = 0 in the third quadrant are not image points of K.

In addition, we might ask which point of K has been mapped to (20,20)?

ie. $20 = \dfrac{10x^2}{x^2+1}$, $20 = \dfrac{10y^2}{y^2+1}$

Taking just the equation in x, we obtain

$$20(x^2+1) = 10x^2$$
$$20x^2 + 20 = 10x^2$$
$$10x^2 = -20$$
$$x^2 = -2$$
$$x = \sqrt{-2}$$ which is not a real number.

From all this we may conclude that the image of K is a portion of the line x - y = 0, running from (0,0) to (10,10) , which includes the origin but **excludes the point (10,10)**.

There is a difference between f(K) and the line x - y = 0.

How do we know when a set of points is a line ?

A set of collinear points like { (0,0) , (1,1) , (2,2) } is not a line. The points may belong to a line. This line has constant slope. If we calculate the slope of the line joining any two points of our set we will get the same answer. Still our set is not a line.

The property we are trying to get hold of is a sort of glued-together-ness of the points which make up a line. The set of real numbers **R** has this property. Indeed, the usual picture we draw to show **R** is such a line - the real number line. There are no gaps in this line and all the strange real numbers such as $\sqrt{2}$, π , e have their place on the line. If we can show a one -to -one link (correspondence) between a set of distinct, collinear points and **R**, then we will have shown that our set is a line.

This could be done quite easily if we could find a way of describing each point in our set by a unique real number. That is, for each real number there is a unique point of the set of collinear points, and for each point of the set there is a unique real number.

The **parametric equation** of a straight line does this precisely. Each point on the line is associated with a unique value of the parameter t , and t is an element of **R.** Contrariwise, for each value of t there corresponds a unique point on the line.

Parametric equations of a straight line

Consider the straight line L : $lx + my + n = 0$

If the point p_1 with co-ordinates (x_1, y_1) and the point p_2 with co-ordinates (x_2, y_2) belong to L, then L may be written in parametric form as the set of points (x, y) such that

$$x = x_1 + t(x_2 - x_1) \quad , \quad y = y_1 + t(y_2 - y_1) \qquad t \in \mathbf{R}. \qquad ....(*)$$

There are a variety of ways of introducing this approach.

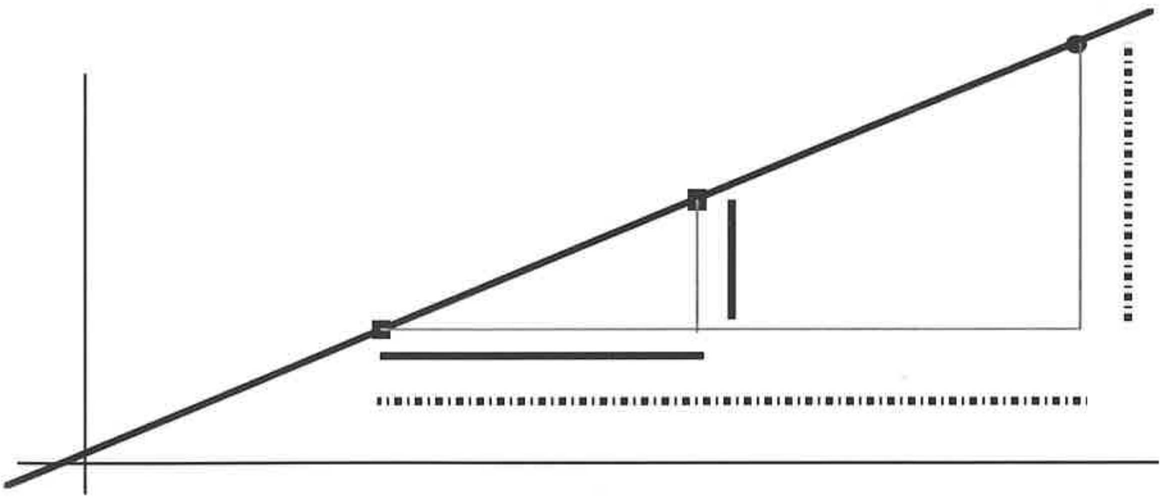
Recall the equation of a straight line in the form $y - y_1 = (slope)(x - x_1)$

and that $slope = \dfrac{y_2 - y_1}{x_2 - x_1}$

hence $y - y_1 = \dfrac{y_2 - y_1}{x_2 - x_1}(x - x_1)$

hence $\dfrac{y - y_1}{y_2 - y_1} = \dfrac{x - x_1}{x_2 - x_1}$ .....(**)

which gives the equation of a line, given two points (x_1, y_1) and (x_2, y_2) on the line.



Consider any point p(x,y) on the line through (x_1,y_1) and (x_2,y_2). The co-ordinates (x,y) will satisfy the equations given in (**).

It is interesting to examine the meaning of the ratio given on each side of the equation (**).

As is shown in the diagram, we may choose to read (**) as a statement about the ratio of sides in two similar triangles. As (x_1,y_1) and (x_2,y_2) are fixed points, the lengths of the lines marked with a ▬▬▬ are fixed numbers. As p moves along the line, the lengths of the lines marked with a ▪▪▪▪▪ change.

If the ratio of the vertical broken line to the vertical solid line is say, 2:1, then the ratio of the corresponding horizontal lines is also 2:1.

More importantly for our purposes, this ratio of 2 can be seen to uniquely define the point p. By choosing a different ratio, say t:1, we can vary p as we wish. If $t \varepsilon \mathbf{R}$, then we ensure that p varies over the entire unbroken line. This use of the ratio t as a parameter provides the form shown in (*):

Let $\dfrac{y - y_1}{y_2 - y_1} = \dfrac{x - x_1}{x_2 - x_1} = t, \quad t \in \mathbf{R}.$

Taking the expression in y first :

$$\frac{y - y_1}{y_2 - y_1} = t \quad \Rightarrow \quad y - y_1 = t(y_2 - y_1)$$

$$\Rightarrow \quad y = y_1 + t(y_2 - y_1)$$

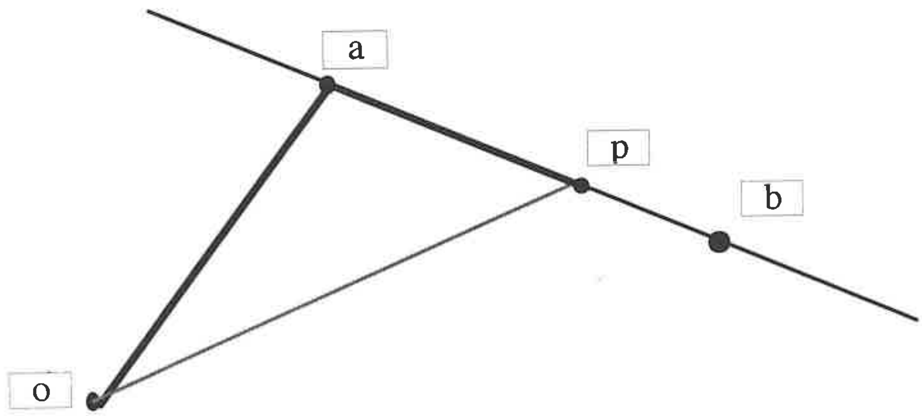
Similarly,

$$\frac{x - x_1}{x_2 - x_1} = t \quad \Rightarrow \quad x - x_1 = t(x_2 - x_1)$$

$$\Rightarrow \quad x = x_1 + t(x_2 - x_1)$$

Giving the form of the parametric equations shown in (*).

A simpler motivating idea is to use a vector approach.



Any point p on the line ab can be expressed as the vector **p** , where

$$\mathbf{p} = \mathbf{a} + t\,\mathbf{ab}, \quad t \varepsilon \mathbf{R} \ . \ \ldots\ldots\ldots\ldots\ldots(\&)$$

If p lies between a and b, then $0 \le t \le 1$.

If p lies "beyond" b, then $t > 1$.

If lies "before" a, then $t < 0$.

If o is the origin $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and a is $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and b is $\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$

then **ab** = **b** - **a** = $\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} - \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \end{pmatrix}$

Thus equation (&)

$$\mathbf{p} = \mathbf{a} + t\mathbf{ab}$$

where $\mathbf{p} = \begin{pmatrix} x \\ y \end{pmatrix}$

can be written as

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + t\begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \end{pmatrix}$$

giving the equations

$x = x_1 + t(x_2 - x_1)$ and $y = y_1 + t(y_2 - y_1)$, $t \in \mathbf{R}$ as in (*).

Let us be quite clear what we have established.

In order to ensure that a set of points p(x,y) is a line, we required a 1-1 correspondence between the set of points and the real numbers **R**.

The claim here is that the parameter tε**R** provides such a link.

Firstly, assume that a value of t, say t= α determines **two** points p and q with co-ordinates

(x_p, y_p) and (x_q, y_q) respectively.

Then from (*), the parametric equations, we have :

$x_p = x_1 + \alpha(x_2 - x_1)$ and $x_q = x_1 + \alpha(x_2 - x_1) \Rightarrow x_p = x_q$

Similarly,

$y_p = y_1 + \alpha(y_2 - y_1)$ and $y_q = y_1 + \alpha(y_2 - y_1) \Rightarrow y_p = y_q$

From which we conclude that the points p and q are the same point.

That is, each value of tε**R** is associated with one unique point on the line.

Secondly, assume that a single point p(x,y) is associated with two values of t, say t=α and t=β.

From the parametric equations (*) we have:

$x = x_1 + \alpha(x_2 - x_1)$ and $x = x_1 + \beta(x_2 - x_1) \Rightarrow \alpha = \beta$

or

$y = y_1 + \alpha(y_2 - y_1)$ and $y = y_1 + \beta(y_2 - y_1) \Rightarrow \alpha = \beta$

From which we conclude that each point is associated with one unique value of tε**R**.

So there is a 1-1 correspondence between the set of points described by the parametric equations of the line and the set of real numbers **R**. **Consequently, we may say that each set of points whose parametric equations are of the form (*) is a line.**

Line segments

Let the symbol **[rs]** indicate a line segment on L, the segment to contain its end points r and s.

All the points of [rs] also belong to L, so they may be written in parametric form.

If the co-ordinates of r are (x_1,y_1) and the co-ordinates of s are (x_2,y_2) then the parametric equations of the line **rs** are

$$x = x_1 + t(x_2 - x_1) \quad , \quad y = y_1 + t(y_2 - y_1) \qquad t \in \mathbf{R}.$$

If we restrict the range of values of the parameter t to $0 \leq t \leq 1$, tε**R** , we neatly obtain the parametric form of the line segment **[rs]**:

$$x = x_1 + t(x_2 - x_1) \quad , \quad y = y_1 + t(y_2 - y_1) \ , \ 0 \leq t \leq 1 \ , t \in \mathbf{R}.$$

Examples:

1. $p(-3,5)$ and $q(6,-4)$ are points. Find parametric equations of the line pq.

Solution:

$x_1 = -3$ and $y_1 = 5$, $x_2 = 6$ and $y_2 = -4$

so

$x = x_1 + t(x_2 - x_1) \Rightarrow x = -3 + t(6 - -3) \Rightarrow x = -3 + 9t$

$y = y_1 + t(y_2 - y_1) \Rightarrow y = 5 + t(-4 - 5) \Rightarrow y = 5 - 9t$

where $t \in \mathbf{R}$.

$x = -3 + 9t$
$y = 5 - 9t$

2. The line K has the equation $3x + 2y - 6 = 0$.
Find parametric equations for K.

Solution :
Obtain two points on K.
If $x = 0$ then $y = 3$ giving $(0,3)$.
If $y = 0$ then $x = 2$ giving $(2,0)$.
$x_1 = 0$ and $y_1 = 3$, $x_2 = 2$ and $y_2 = 0$
so
$x = x_1 + t(x_2 - x_1) \Rightarrow x = 0 + t(2 - 0) \Rightarrow x = 2t$
$y = y_1 + t(y_2 - y_1) \Rightarrow y = 3 + t(0 - 3) \Rightarrow y = 3 - 3t$
where $t \in \mathbf{R}$.

$$x = 2t$$
$$y = 3 - 3t$$

3. Given $r(-8,4)$ and $s(1,8)$, obtain parametric equations for the line segment [rs].
Solution :
$x_1 = -8$ and $y_1 = 4$, $x_2 = 1$ and $y_2 = 8$
so
$x = x_1 + t(x_2 - x_1) \Rightarrow x = -8 + t(1 - -8) \Rightarrow x = -8 + 9t$
$y = y_1 + t(y_2 - y_1) \Rightarrow y = 4 + t(8 - 4) \Rightarrow y = 4 + 4t$
where $0 \leq t \leq 1$, $t \in \mathbf{R}$.
Check : if $t = 0$ then $x = -8$ and $y = 4$
if $t = 1$ then $x = 1$ and $y = 8$.

$$x = -8 + 9t$$
$$y = 4 + 4t$$

NOTE:

A useful property of such parametric equations of line segments is that the co-ordinates of a point w which divides [rs] internally in the ratio of say 3:4 can be quickly found by letting t=3/7 , or more generally, if the ratio is a:b, letting t=a/(a+b). The external case is dealt with as follows: if a point divides [rs] externally in the ratio of a:b, then the point is obtained by letting t=a/(a-b).

4. As one final step towards our four theorems, we now find the image of the line K : $5x-2y+10 = 0$ (say) under the transformation **f** we have used in our examples,

where **f** : $(x,y) \rightarrow (x',y')$ and
where $x' = 3x - y$
$y' = x + 2y$

The first step is to express the line K in parametric form:

when x = 0, y = 5 giving (0,5)
when y = 0, x = -2 giving (-2,0)
$x_1 = 0$ and $y_1 = 5$, $x_2 = -2$ and $y_2 = 0$
so
$x = x_1 + t(x_2 - x_1) \Rightarrow x = 0 + t(-2 - 0) \Rightarrow x = -2t$
$y = y_1 + t(y_2 - y_1) \Rightarrow y = 5 + t(0 - 5) \Rightarrow y = 5 - 5t$
where $t \in \mathbf{R}$.

$x = -2t$
$y = 5 - 5t$

We now apply the transformation **f**:

$x' = 3x - y \Rightarrow x' = 3(-2t) - (5 - 5t) \Rightarrow x' = -5 - t$
$y' = x + 2y \Rightarrow x' = (-2t) + 2(5 - 5t) \Rightarrow y' = 10 - 12t$
where $t \in \mathbf{R}$.
These are parametric equations of a line, so K has been mapped under **f** to a line.
The standard equation of $\mathbf{f}(\mathrm{K})$ may be obtained as follows:
$x' = -5 - t \Rightarrow t = -5 - x'$
$y' = 10 - 12\mathrm{t} \Rightarrow y' = 10 - 12(-5 - x')$
$\Rightarrow y' = 70 + 12x'$
$\Rightarrow 12x' - y' + 70 = 0$ is the equation of **f**(K).

Introduction to the Theorems

In the following four 'theorems', we will use a specific transformation of the type set out in the syllabus. All such transformations are well-behaved in that a line will be mapped to a line, a line segment to a line segment, a pair of parallel lines to a pair of parallel lines and each parallelogram to a parallelogram. (Full proof in the OPTION).

The transformation is **f**
where **f** : $(x,y) \rightarrow (x',y')$ and
where $x' = 3x - y$
$y' = x + 2y$

(NB : the determinant of the corresponding matrix is non-zero - what would it mean if the determinant was 0 ?)

Theorem 1.

f maps each line to a line

Proof:

Let L be any line $lx + my + n = 0$ where **neither *l* nor *m* is zero.**

Choose any two points of L .

Let x = 0 , so y = $-\frac{n}{m}$ giving $\left(0, -\frac{n}{m}\right)$

and y = 0, so x = $-\frac{n}{l}$ giving $\left(-\frac{n}{l}, 0\right)$.

The parametric equations of L are now

$$x = 0 + t\left(-\frac{n}{l} - 0\right) = -\frac{tn}{l}$$

$$y = -\frac{n}{m} + t\left(0 - -\frac{n}{m}\right) = -\frac{n}{m} + \frac{tn}{m}$$

where $t \in \mathbf{R}$.

$$x = -\frac{tn}{l}$$
$$y = -\frac{n}{m} + \frac{tn}{m}$$

Applying the transformation **f**:

$$x' = 3\left(-\frac{tn}{l}\right) + \frac{n}{m} - \frac{tn}{m} = \frac{n}{m} + t\left(-\frac{3n}{l} - \frac{n}{m}\right)$$

$$y' = -\frac{tn}{l} - \frac{2n}{m} + \frac{2tn}{m} = -\frac{2n}{m} + t\left(\frac{2n}{m} - \frac{n}{l}\right)$$

that is, the parametric equations of the image of L under **f** have the form

$x' = p + tq$

$y' = r + ts$

where $t \in \mathbf{R}$.

We conclude therefore that **f**(L) is a line. QED.

Theorem 2. **f** maps each line segment to a line segment.

Proof:

Consider the line segment [rs] with $r(x_1, y_1)$ and $s(x_2, y_2)$ two distinct points.
The parametric equations of [rs] are

$$x = x_1 + t(x_2 - x_1)$$
$$y = y_1 + t(y_2 - y_1)$$

with $0 \le t \le 1, t \in \mathbf{R}$.

Applying **f** :

$$x' = 3[x_1 + t(x_2 - x_1)] - [y_1 + t(y_2 - y_1)] = (3x_1 - y_1) + t(3x_2 - 3x_1 - y_2 + y_1).$$
$$y' = x_1 + t(x_2 - x_1) + 2[y_1 + t(y_2 - y_1)] = x_1 + 2y_1 + t(x_2 - x_1 + 2y_2 - 2y_1).$$

With $0 \le t \le 1, t \in \mathbf{R}$.

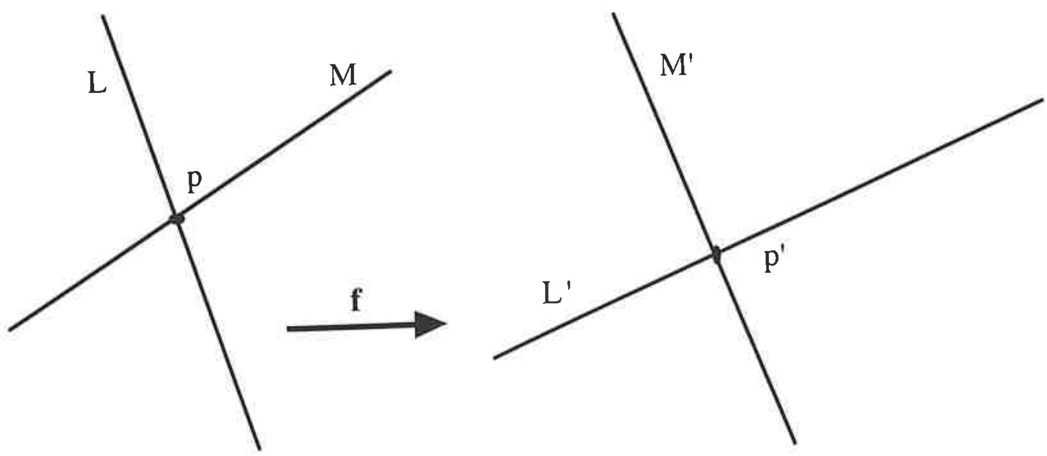
These are the parametric equations of the image of [rs] under **f** and conform to the parametric equations of a line segment beginning at the point obtained by letting $t = 0$ and ending at the point obtained by letting $t = 1$. QED.

Theorem 3

Let L and M be lines, such that L and M meet at some point p.

Applying a transformation **f**, and using **Theorem 1**, L is mapped to a line L' = f(L), and M is mapped to a line M' = f(M). Let p be mapped to a point p'=f(p).

As $p \in L$ then $p' \in L'$.
Similarly, as $p \in M$ then $p' \in M'$.
So $p' \in L' \cap M'$.
Thus the lines L' and M' meet at p' .



Now let L || M .

Either L=M, in which case L'=M',

or $L \cap M = \varnothing$.

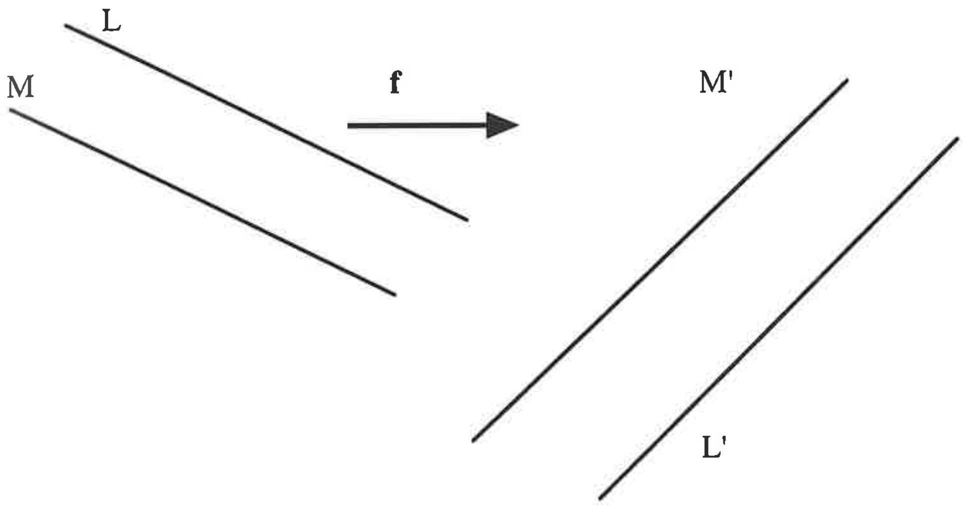
If the latter, then $L' \cap M' = \varnothing$, which implies $L' \parallel M'$.

Otherwise, if $L' \cap M' = q$ then the inverse map $\mathbf{f}^{-1}$

would map q to some point $\mathbf{f}^{-1}(q)$ such that $\mathbf{f}^{-1}(q) \in L \cap M$,

which contradicts $L \cap M = \varnothing$.

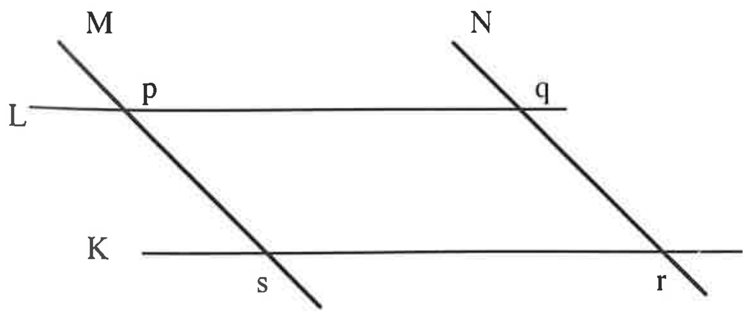
QED.



<u>**Theorem 4.**</u> **f** maps each parallelogram to a parallelogram

Proof:

We assume that **f** maps each line to a line, and each pair of parallel lines to a pair of parallel lines.



Let L, K, M, and N be distinct lines.
Let L || K and M || N , and let *pqrs* be a parallelogram.

f will map the lines L,K,M,and N to lines **f**(L),**f**(K),**f**(M) and **f**(N).

f will map the pair of parallel lines L,K to a pair of parallel lines **f**(L),**f**(K).

f will map the pair of parallel lines M,N to a pair of parallel lines **f**(M),**f**(N).

$p \in M \cap L \Rightarrow \mathbf{f}(p) \in \mathbf{f}(M) \cap \mathbf{f}(L)$

$q \in N \cap L \Rightarrow \mathbf{f}(q) \in \mathbf{f}(N) \cap \mathbf{f}(L)$

$r \in K \cap N \Rightarrow \mathbf{f}(r) \in \mathbf{f}(K) \cap \mathbf{f}(N)$

$s \in K \cap M \Rightarrow \mathbf{f}(s) \in \mathbf{f}(K) \cap \mathbf{f}(M)$

$\Rightarrow \mathbf{f}(p)\mathbf{f}(q)\mathbf{f}(r)\mathbf{f}(s)$ is a parallelogram. QED.

Appendix

Alternative proofs:

Theorem 1 **f** maps a line to a line

Let L be the line $lx + my + n = 0$.

Now **f** is the transformation represented by the matrix $\begin{pmatrix} 3 & -1 \\ 1 & 2 \end{pmatrix}$

such that $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 3 & -1 \\ 1 & 2 \end{pmatrix}^{-1}\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ 1 & 2 \end{pmatrix}^{-1}\begin{pmatrix} 3 & -1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\Rightarrow \frac{1}{7}\begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \frac{2}{7} & \frac{1}{7} \\ \frac{-1}{7} & \frac{3}{7} \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\Rightarrow \frac{2}{7}x' + \frac{1}{7}y' = x \quad \text{and} \quad -\frac{1}{7}x' + \frac{3}{7}y' = y$$

Applying the transformation **f** to L: $lx + my + n = 0$,

we find that the set of points **f**(L) satisfy the equation

$$l\left(\frac{2}{7}x' + \frac{1}{7}y'\right) + m\left(-\frac{1}{7}x' + \frac{3}{7}y'\right) + n = 0$$

or $$\left(\frac{2l - m}{7}\right)x' + \left(\frac{l + 3m}{7}\right)y' + n = 0 \qquad ....(***)$$

Consider the **line** whose equation is (***).

Let a point of this line be (x', y').

Then there exists a transformation given by the equations

$x' = 3x - y$, $y' = x + 2y$ which maps (x', y') as follows:

$$\left(\frac{2l - m}{7}\right)(3x - y) + \left(\frac{l + 3m}{7}\right)(x + 2y) + n = 0$$

$$\Rightarrow (2l - m)(3x - y) + (l + 3m)(x + 2y) + 7n = 0$$

$$\Rightarrow 6lx - 2ly - 3mx + my + lx + 2ly + 3mx + 6my + 7n = 0$$

$$\Rightarrow 7lx + 7my + 7n = 0$$

$$\Rightarrow lx + my + n = 0$$

that is, each point of the **line** $\left(\frac{2l - m}{7}\right)x' + \left(\frac{l + 3m}{7}\right)y' + n = 0$

may be mapped to the line L: $lx + my + n = 0$.

Thus **f** maps each line to a line.

QED.

Theorem 3. **f** maps each pair of parallel lines to a pair of parallel lines

Proof:

We assume that **f** maps each line to a line.

Let L and K be two parallel lines.

If L = K, then **f**(L) = **f**(K) and **f**(K) ∥ **f**(L).

Otherwise, let L : $lx + my + n = 0$ and K : $lx + my + v = 0$.

Now **f** is the transformation represented by the matrix $\begin{pmatrix} 3 & -1 \\ 1 & 2 \end{pmatrix}$

such that $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 3 & -1 \\ 1 & 2 \end{pmatrix}^{-1}\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ 1 & 2 \end{pmatrix}^{-1}\begin{pmatrix} 3 & -1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\Rightarrow \frac{1}{7}\begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \frac{2}{7} & \frac{1}{7} \\ \frac{-1}{7} & \frac{3}{7} \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\Rightarrow \frac{2}{7}x' + \frac{1}{7}y' = x \quad \text{and} \quad -\frac{1}{7}x' + \frac{3}{7}y' = y$$

Applying **f** to the line L: $lx + my + n = 0$

we obtain $l\left(\frac{2}{7}x' + \frac{1}{7}y'\right) + m\left(-\frac{1}{7}x' + \frac{3}{7}y'\right) + n = 0$

Applying **f** to the line K: $lx + my + v = 0$

we obtain $l\left(\frac{2}{7}x' + \frac{1}{7}y'\right) + m\left(-\frac{1}{7}x' + \frac{3}{7}y'\right) + v = 0$

Without further rearrangement we can see that the slope of **f**(L) and the slope of **f**(K) will be equal.

As **f**(L) are **f**(K) are lines, then **f**(L) || **f**(K) QED.

OR using parametric equations

Theorem 3 f maps each pair of parallel lines to a pair of parallel lines

Proof
Assume that **f** maps each line to a line.
Let L be any line $lx + my + n = 0$ where **neither *l* nor *m* is zero**.
Let K be any line $lx + my + v = 0$ where **neither *l* nor *m* is zero**.

That is, L and K are a pair of parallel lines.

If L=K, then **f**(L)=**f**(K) and **f**(L) || **f**(K) .

Otherwise. let L and K be distinct parallel lines.

Choose any two points of L .

Let x = 0 , so y = $-\frac{n}{m}$ giving $\left(0,-\frac{n}{m}\right)$

and y = 0, so x = $-\frac{n}{l}$ giving $\left(-\frac{n}{l},0\right)$.

The parametric equations of L are now

$$x = 0 + t\left(-\frac{n}{l} - 0\right) = -\frac{tn}{l}$$

$$y = -\frac{n}{m} + t\left(0 - -\frac{n}{m}\right) = -\frac{n}{m} + \frac{tn}{m}$$

where $t \in \mathbf{R}$.

$$x = -\frac{tn}{l}$$
$$y = -\frac{n}{m} + \frac{tn}{m}$$

Applying the transformation **f** :

$$x' = 3\left(-\frac{tn}{l}\right) + \frac{n}{m} - \frac{tn}{m} = \frac{n}{m} + t\left(-\frac{3n}{l} - \frac{n}{m}\right)$$

$$y' = -\frac{tn}{l} - \frac{2n}{m} + \frac{2tn}{m} = -\frac{2n}{m} + t\left(\frac{2n}{m} - \frac{n}{l}\right)$$

which are the parametric equations of the line **f**(L).

In order to obtain the slope of **f**(L), we let t=0 and t=1 to obtain two points with which to calculate the slope.

If t = 0 we obtain $\left(\frac{n}{m},-\frac{2n}{m}\right)$

If t = 1 we obtain $\left(-\frac{3n}{l},-\frac{n}{l}\right)$.

Applying the slope formula, we obtain the slope as

$$\frac{-\frac{2n}{m}+\frac{n}{l}}{\frac{n}{m}+\frac{3n}{l}}=\frac{-\frac{2}{m}+\frac{1}{l}}{\frac{1}{m}+\frac{3}{l}} \quad \text{(the } n \text{ cancels out)}$$

$$\text{so the slope of } \mathbf{f}(\text{L}) = \frac{m-2l}{3m+l}$$

Repeating the same process for K, we obtain parametric equations for **f**(K) and obtain for the slope of the line **f**(K) the expression:

$$\frac{-\frac{2v}{m}+\frac{v}{l}}{\frac{v}{m}+\frac{3v}{l}}=\frac{-\frac{2}{m}+\frac{1}{l}}{\frac{1}{m}+\frac{3}{l}} \quad \text{(the } v \text{ cancels out)}$$

$$\text{so the slope of } \mathbf{f}(\text{K}) = \frac{m-2l}{3m+l}$$

that is, the line **f**(L) is parallel to the line **f**(K). QED.

<u>NOTE</u>: in the proofs of some of these theorems we used a technique from matrix algebra to obtain from the transformation equations expressions for x in terms of x' and y', and for y in terms of x' and y'. We then substituted into the equations of the lines L and K to obtain the equations of their images under **f**.

Why can this technique not be used all the time?

The answer is that using such a method, we obtain an equation which the image points of the line must satisfy. However, this does not prove that the image of the line **is a line**, and not simply a set of collinear points. To show this, we must demonstrate that we can map each element of the **line** whose equation we have obtained back to our original line. (Look back at the section called A Pathological Case - we tried to map the point (20,20) back to our original line and couldn't do it. In this way we demonstrated that the image of the line x+y=0 under **f** is not a line).

We are permitted to use this technique in Theorem 3 because we have proved in Theorem 2 that the transformation in question maps a line to a line. Such an assumption is surely permitted in the examination, otherwise the students are being asked to prove TWO theorems!

Solution to Leaving Certificate Question

Leaving Cert. 1995 Paper II Q 3(b) and Q3(c)

(b)

f is the transformation $(x,y) \to (x',y')$ where $x' = 3x - y$ $y' = x + 2y$

For points $p(0,0)$, $q(1,0)$ and $r(0,2)$ find $f(p)$, $f(q)$, and $f(r)$.

Investigate if (i) $|qr| = |f(q)f(r)|$

(ii) the area of the triangle pqr is equal to thge area of the triangle $f(p)f(q)f(r)$

(c)

L is a line with equation $ax + by + c = 0$. Prove that the image $f(L)$ is also a line, where f is the transformation in (b).

M is a line with equation $ax + by + d = 0$. Deduce the equation of $f(M)$.

Show $f(L) \parallel f(M)$.

Solution **(b)**

$$\begin{pmatrix} 3 & -1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 3 & -2 \\ 0 & 1 & 4 \end{pmatrix} \Rightarrow f(p)(0,0),\ f(q)(3,1) \text{ and } f(r)(-2,4)$$

$|qr| = \sqrt{5}$ $|f(q)f(r)| = \sqrt{34}$ $\Rightarrow$ not equal

area $\Delta\, prq$ = 1 unit^2 and area $\Delta\ f(p)f(q)f(r)$ = 7 units^2

[using $\frac{1}{2}|x_2y_1 - x_1y_2|$]

$\Rightarrow$ not equal

(NOTE: the determinant of the transformation is 7 - a useful check here).

(c) Proof:

Let L be any line $ax + by + c = 0$ where neither a nor b is zero.

Choose any two points of L .

Let x = 0 , so y = $-\frac{c}{b}$ giving $\left(0,-\frac{c}{b}\right)$

and y = 0, so x = $-\frac{c}{a}$ giving $\left(-\frac{c}{a},0\right)$.

The parametric equations of L are now

$$x = 0 + t\left(-\frac{c}{b} - 0\right) = -\frac{tc}{a}$$

$$y = -\frac{c}{b} + t\left(0 - -\frac{c}{b}\right) = -\frac{c}{b} + \frac{tc}{b}$$

where $t \in \mathbf{R}$.

Applying the transformation **f**:

$$x' = 3\left(-\frac{tc}{a}\right) - \left(-\frac{c}{b} + \frac{tc}{b}\right) = \frac{c}{b} + t\left(-\frac{3c}{a} - \frac{c}{b}\right)$$

$$y' = -\frac{tc}{a} + 2\left(-\frac{c}{b} + \frac{tc}{b}\right) = -\frac{2c}{b} + t\left(-\frac{c}{a} + \frac{2c}{b}\right)$$

that is, the parametric equations of the image of L under **f**

have the form

$x' = p + tq$

$y' = r + ts$

where $t \in \mathbf{R}$.

We conclude therefore that **f**(L) is a line. QED.

By examination of the foregoing, we deduce that $f(M)$ is also a line, with parametric equations

$$x' = \frac{d}{b} + t\left(-\frac{3d}{a} - \frac{d}{b}\right)$$

$$y' = -\frac{2d}{b} + t\left(-\frac{d}{a} + \frac{2d}{b}\right)$$

We can obtain two points on $f(L)$ and calculate its slope.

We can do the same for $f(M)$ and then compare answers to see if the two image lines are parallel.

We obtain the slope of each line to be $\dfrac{2a-b}{-a-3b} \Rightarrow f(L) \parallel f(M)$

Note on the use of parametric equations

Students who have been prepared using the parametric approach would need to know how to calculate the inverse of a given transformation. Although they do not require this technique for the proofs of the theorems, they need it for more elementary calculations such as the following:

Given the transformation $f\ :\ (x,y) \to (x',y')$ where $x' = 3x - y \quad y' = x + 2y$, find an expression for the inverse transformation f^{-1} , and hence find $f^{-1}(p)$, where p is the point (7,-14).

Solution :

The action of f on a point (x,y) can be represented in matrix form as follows:

$$\begin{pmatrix} 3 & -1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix} \qquad (\&)$$

We calculate the inverse of the matrix $\begin{pmatrix} 3 & -1 \\ 1 & 2 \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}$ and multiply (LHS) each side of (&) to get

$$\frac{1}{7}\begin{pmatrix}2 & 1\\ -1 & 3\end{pmatrix}\begin{pmatrix}3 & -1\\ 1 & 2\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \frac{1}{7}\begin{pmatrix}2 & 1\\ -1 & 3\end{pmatrix}\begin{pmatrix}x'\\ y'\end{pmatrix}$$

giving $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}\frac{2}{7} & \frac{1}{7}\\ -\frac{1}{7} & \frac{3}{7}\end{pmatrix}\begin{pmatrix}x'\\ y'\end{pmatrix} \Rightarrow \begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}\frac{2}{7}x' + \frac{1}{7}y'\\ -\frac{1}{7}x' + \frac{3}{7}y'\end{pmatrix}$

$\Rightarrow$ $x = \frac{2}{7}x' + \frac{1}{7}y'$ and $y = -\frac{1}{7}x' + \frac{3}{7}y'$

showing how f^{-1} maps $(x',y') \to (x,y)$

Hence $f^{-1}(p)$ is found through calculating (x,y) by substituting $x' = 7$ and $y' = -14$ to obtain $f^{-1}(p)$ as $(0,-7)$

FURTHER GEOMETRY

Consider the following problem :

prove that the tangents at the endpoints of a diameter of an ellipse are parallel.

Assume that it has already been established that

(i) there is a transformation **f** that maps a unit circle, centre (0,0) to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

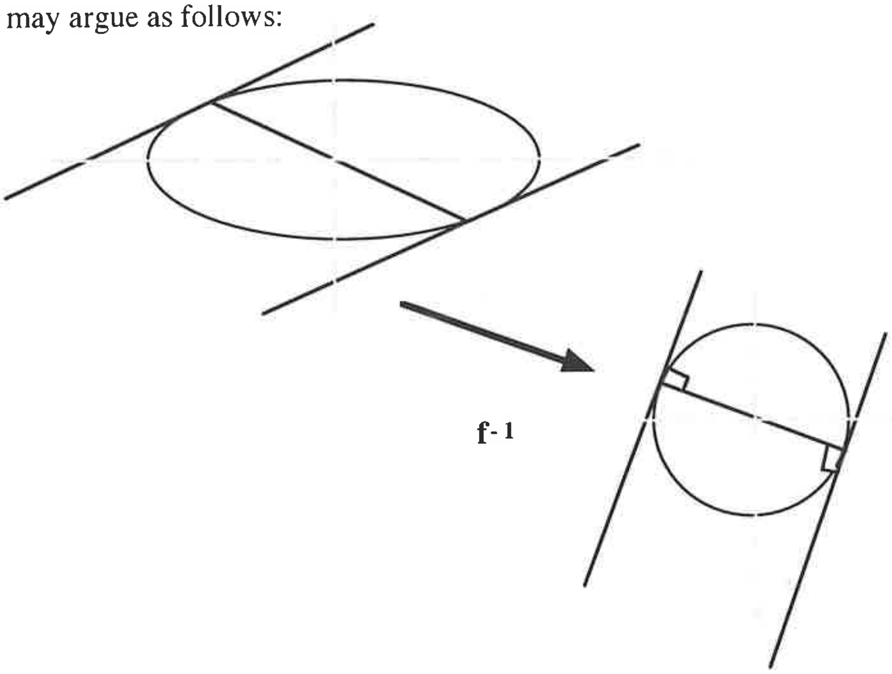
and a transformation $\mathbf{f}^{-1}$ which maps this ellipse back to the circle

(ii) the transformation $\mathbf{f}^{-1}$ maps any diameter of the ellipse to a diameter of the circle, and **f** maps any diameter of the circle to a diameter of the ellipse

(iii) the transformations **f** and $\mathbf{f}^{-1}$ map pairs of parallel lines to pairs of parallel lines.

(iv) the transformations **f** and $\mathbf{f}^{-1}$ map a tangent to a tangent

Then we may argue as follows:



The diameter (and endpoints) of the ellipse is mapped under $\mathbf{f}^{-1}$ to a diameter (and endpoints) of the circle. The tangents at the endpoints of the diameter of the ellipse are mapped to the tangents at the endpoints of the diameter of the circle.

The tangents at the endpoints of the diameter of a circle are parallel.

The transformation **f** maps these tangents to the circle back to the original tangents to the ellipse.

The transformation **f** maps pairs of parallel lines to pairs of parallel lines.
Consequently, we may deduce that tangents at the endpoints of a diameter of an ellipse are parallel.

The syllabus describes this approach as "deduction from results for a circle of similar results for an ellipse". The results deal with

(a) the centre of an ellipse

(b) tangents at the endpoints of a diameter of an ellipse

(c) locus of midpoints of parallel chords of an ellipse

(d) locus of harmonic conjugates of a point with respect to an ellipse (pole and polar)

(e) areas of all parallelograms circumscribed to an ellipse at the endpoints of conjugate diameters.

In setting out deduction (b) it was assumed that three items had already been established. The first item concerned the transformation **f** that maps a unit circle, centre (0,0) to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

This transformation **f** is simply

x'	=	ax	with	x	=	x'/a
y'	=	by		y	=	y'/b

and substituting for x and y in

$$x^2+y^2=1$$

gives

$$\frac{(x')^2}{a^2}+\frac{(y')^2}{b^2}=1$$

The transformation $\mathbf{f}^{-1}$ is

x'	=	x/a	with	x	=	ax'
y'	=	y/b		y	=	bx'

and substituting for x and y in

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

gives

$$(x')^2+(y')^2=1$$

as required.

The second and third items assumed to have been established concerned properties (being a diameter, being a pair of parallel lines) that were carried over intact by **f** and $\mathbf{f}^{-1}$. That is, properties that were <u>invariant</u> under either transformation.

One way of proceeding to deduce the results (a) to (e) is to thoroughly examine which properties (in particular, the ones we need for our deductions) are invariant under the given transformations **f** and $\mathbf{f}^{-1}$.

Alternatively, we may consider which geometrical properties are invariant under a general type of transformation, the **affine transformation** of which **f** and $\mathbf{f}^{-1}$ happen to be examples.

It is this latter route which is indicated by the syllabus:

> Transformations **f** of the plane Π which have the co-ordinate form
>
> $$(x,y) \to (x',y') \text{ where :}$$
>
> $$x' = ax + by + k_1$$
> $$y' = cx + dy + k_2$$
>
> and $ad - bc \neq 0$.
>
> Use of matrices.
>
> Magnification ratio. Invariance of ratio of lengths on parallel lines. Invariance of centroid of a triangle.
>
> Invariance of ratio of areas.

It may be useful to consider what work the syllabus sets out for the <u>core</u> Transformation Geometry:

> Each transformation **f** of the plane Π which has the coordinate form
>
> $$(x,y) \to (x',y') \quad \text{where}$$
>
> $$x' = ax + by$$
> $$y' = cx + dy$$
>
> and $ad - bc \neq 0$,
>
> maps each line to a line, each line segment to a line segment, each pair of parallel lines to a pair of parallel lines, and consequently each parallelogram to a parallelogram.
> *Proof confined to a specific transformation (numerical values for a, b, c, and d) .*
>
> Examples of the invariance or non-invariance of perpendicularity, distance, ratio of two distances, area, and ratio of two areas connected with specific parallelograms (including rectangles and squares) under transformations of the form
>
> $$x' = ax + by$$
> $$y' = cx + dy$$
>
> with numerical coefficients.

In making the transition from the core material to the optional material the students will be aware of the more abstract style of work. As an introduction to the new material, it is important that students be given an opportunity to do some investigatory work with a specific affine transformation such as

$$x' = 2x + 3y + 4$$
$$y' = 4x - 5y + 2$$

and its matrix form

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 4 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 4 \\ 2 \end{pmatrix}$$

so that the introduction of the new element of the transformation, the addition of a vector, can be seen to correspond to a translation.

It might be pointed out that the core transformations are affine transformations of type

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{with } ad - bc \neq 0$$

and that the reason why a translation cannot be expressed in such a form is, of course, that the origin is invariant under this type of transformation, while a translation (unless the identity translation) will move the origin.

An interesting question now is if there are **any** invariant points under affine transformations such as

$$x' = 2x + 3y + 4$$
$$y' = 4x - 5y + 2$$

that is, when is (x',y') =(x,y) ?

This requires the solution of the simultaneous equations

$$x' = x \quad \text{or} \quad 2x + 3y + 4 = x$$
$$y' = y \quad \text{or} \quad 4x - 5y + 2 = y$$

which simplifies to

$$x + 3y = -4$$
$$4x - 6y = -2$$

giving the solution $\left(-\frac{5}{3} \ , \ -\frac{7}{9} \right)$ as the invariant point.

A useful result for the purposes of constructing questions with reasonable answers is that if

$$x' = ax + by + k_1$$
$$y' = cx + dy + k_2$$

and $ad - bc \neq 0$

then the invariant point (x,y) has coordinates

$$x = \frac{-k_1(d-1) + bk_2}{(a-1)(d-1) - bc} \quad , \quad y = \frac{-k_2(a-1) + ck_1}{(a-1)(d-1) - bc}$$

providing that $(a-1)(d-1) - bc \neq 0$.

As formal introductions are now called for, let us begin.

1 : Affine Transformations

A function $f : \Pi \to \Pi : p \to p'$ such that if p and p' have coordinates (x,y) , (x',y') , respectively, then

$x' = ax + by + k_1$

$y' = cx + dy + k_2$

for some constants a,b,c,d,k_1,k_2 for which

$ad - bc \neq 0,$

is called an *affine transformation* of Π.

An affine transformation has matrix form

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix} + K,$$

where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad K = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} \text{ and } ad - bc \neq 0 .$$

We denote by Cl the set of all affine transformations of Π .

It is important to establish the result that each affine transformation has an inverse which is also an affine transformation.

Given $\mathbf{f} \in Cl$, with

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix} + K,$$

consider $\mathbf{g}$ defined by

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = A^{-1}\begin{pmatrix} x \\ y \end{pmatrix} - A^{-1}K,$$

where A^{-1} is the inverse matrix of A .

Then $\mathbf{g} \in Cl$ as it has the appropriate form, and as det A $\neq$ 0, so det $A^{-1} \neq 0$.

Recall that if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $\det A = ad - bc$

and $A^{-1} = \dfrac{1}{\det A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

Then $\mathbf{g} \circ \mathbf{f} = \mathbf{f} \circ \mathbf{g} = \mathbf{i}_\Pi$, the identity transformation on the plane Π,

with matrix $\mathbf{i}_\Pi = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

We now establish in general terms some results from the core which were established using specific transformations.

Transformations **f** of the plane $\prod$ which have the co-ordinate form

$$(x,y) \to (x',y') \quad \text{where}$$

$$x' = ax + by + k_1$$
$$y' = cx + dy + k_2$$

$$\text{and} \quad ad - bc \neq 0 \ ,$$

map

each line to a line **1.1**

each line segment to a line segment **1.2**

each pair of parallel lines to a pair of parallel lines **1.3**

We prove **1.1** and **1.2** most easily by using the parametric equation of a line. (See pp. 17 ff. of these notes).

Result **1.3** is proved quite neatly using a set definition of parallel lines.

Proof 1.1

If $p_1 \equiv (x_1, y_1)$, $p_2 \equiv (x_2, y_2)$ are distinct points, the line $L = p_1p_2$ has parametric equations

$x = x_1 + t(x_2 - x_1)$, $\qquad y = y_1 - t(y_2 - y_1)$ where $t \in \mathbf{R}$

and (x,y) is a representative point on the line L.

Using the equations for an affine transformation, we map

$(x,y) \to (x',y')$ where :

$$x' = ax + by + k_1$$
$$y' = cx + dy + k_2$$

and $ad - bc \neq 0$.

So

$$\begin{aligned}
x' &= a\left[x_1 + t(x_2 - x_1)\right] + b\left[y_1 + t(y_2 - y_1)\right] + k_1 \\
&= ax_1 + by_1 + k_1 + at(x_2 - x_1) + bt(y_2 - y_1) \\
&= ax_1 + by_1 + k_1 + t\left[a(x_2 - x_1) + b(y_2 - y_1)\right] && 1.1.1 \\
&= ax_1 + by_1 + k_1 + t\left[(ax_2 + by_2) - (ax_1 + by_1)\right] \\
&= ax_1 + by_1 + k_1 + t\left[(ax_2 + by_2 + k_1) - (ax_1 + by_1 + k_1)\right] && 1.1.2
\end{aligned}$$

Under the affine transformation, the points $p_1 \equiv (x_1,y_1)$ and $p_2 \equiv (x_2,y_2)$, which were given on the line L, are mapped to image points $p_1' \equiv (x_1',y_1')$ and $p_2' \equiv (x_2',y_2')$ as follows:

p_1 : $x_1' = ax_1 + by_1 + k_1$ $\quad y_1' = cx_1 + dy_1 + k_2$

p_2 : $x_2' = ax_2 + by_2 + k_1$ $\quad y_2' = cx_2 + dy_2 + k_2$

Thus 1.1.2 above

$$x' = ax_1 + by_1 + k_1 + t[(ax_2 + by_2 + k_1) - (ax_1 + by_1 + k_1)] \qquad 1.1.2$$

can be re-written as

$x' = x_1' + t(x_2' - x_1')$ where $t \in \mathbf{R}$.

The same procedure applied to y' yields

$y' = y_1' + t(y_2' - y_1')$ where $t \in \mathbf{R}$.

These parametric equations for $p' \equiv (x',y')$ show that the line L is mapped to a line L' . **QED.**

NB the possibility that L is mapped to a point only is covered by the restriction that ad-bc $\neq 0$, for

$$x' = ax_1 + by_1 + k_1 + t[a(x_2 - x_1) + b(y_2 - y_1)] \qquad 1.1.1$$

and the equivalent statement for y'

$$y' = cx_1 + dy_1 + k_2 + t[c(x_2 - x_1) + d(y_2 - y_1)]$$

will only amount to a single point if the contents of the square brackets are both zero.

ie.

$a(x_2 - x_1) + b(y_2 - y_1) = 0$

$c(x_2 - x_1) + d(y_2 - y_1) = 0$

As ad-bc$\neq 0$, the only solutions are $x_1=x_2$ and $y_1=y_2$ which mean that $p_1=p_2$, a contradiction, as the points are distinct.

Proof 1.2

The line segment $[p_1p_2]$ has parametric equations

$x = x_1 + t(x_2 - x_1)$, $\quad y = y_1 - t(y_2 - y_1)$ where $0 \le t \le 1$, $t \in \mathbf{R}$

for a representative point (x,y) on the line segment.

Following from **1.1**, such points (x,y) will be mapped to (x',y') where

$x' = x_1' + t(x_2' - x_1')$, $\quad y' = y_1' + t(y_2' - y_1')$ with $0 \le t \le 1$, $t \in \mathbf{R}$.

That is, the line segment $[p_1'p_2']$. **QED.**

Proof 1.3

Let L and M be lines, such that L and M meet at some point p.

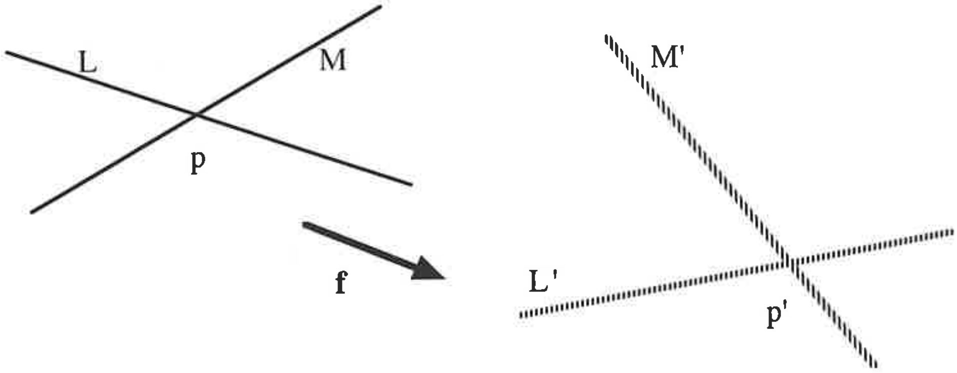
Applying an affine transformation **f**, and using **1.1**, L is mapped to a line L' = f(L), and M is mapped to a line M' = f(M). Let p be mapped to a point p'=f(p).

As $p \in L$ then $p' \in L'$.

Similarly, as $p \in M$ then $p' \in M'$.

So $p' \in L' \cap M'$.

Thus the lines L' and M' meet at p' .



Now let L || M .

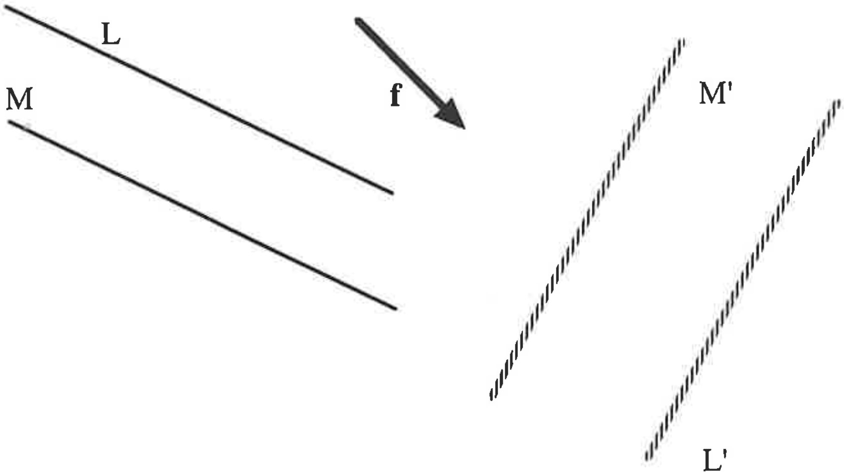
Either L=M, in which case L'=M',

or $L \cap M = \varnothing$.

If the latter, then $L' \cap M' = \varnothing$, which implies $L' \parallel M'$.

Otherwise, if $L' \cap M' = q$ then the inverse map $\mathbf{f}^{-1}$ would map q to some point $\mathbf{f}^{-1}(q)$ such that $\mathbf{f}^{-1}(q) \in L \cap M$, which contradicts $L \cap M = \varnothing$.

QED.



We now establish some new results.

Transformations **f** of the plane $\prod$ which have the co-ordinate form

$$(x,y) \rightarrow (x',y') \quad \text{where}$$

$$x' = ax + by + k_1$$
$$y' = cx + dy + k_2$$

$$\text{and} \quad ad - bc \neq 0 \ ,$$

map

half-planes with a common edge to half-planes with a common edge **1.4**

triangles to triangles **1.5**

parallelograms to parallelograms **1.6**

Proof 1.4

Let H_1 and H_2 be half-planes with common edge L. That is, the union of H_1 and H_2 is the entire plane $\prod$ and the intersection of H_1 and H_2 is the line L.

Let x be a point of H_1 not on the edge L.
Let **f** be an affine transformation. Hence $\mathbf{f^{-1}}$ exists and is an affine transformation.

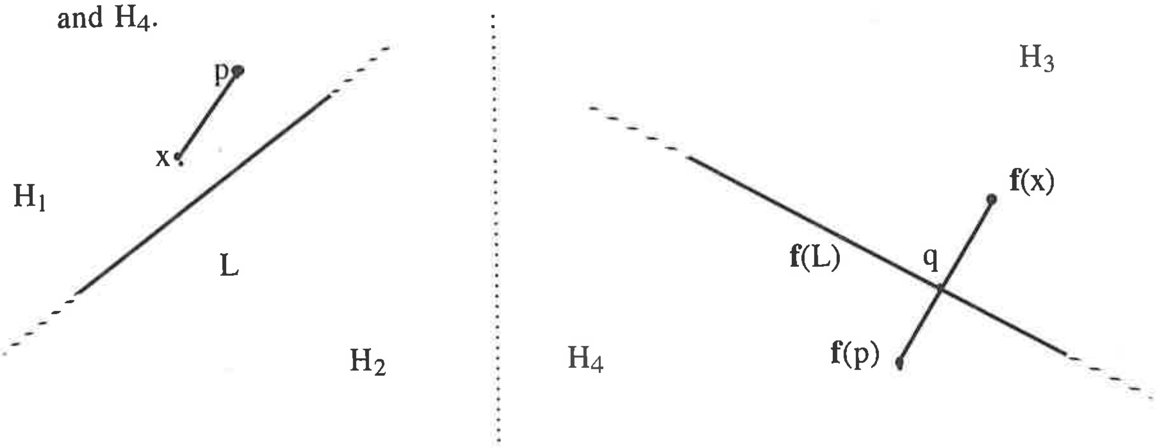
Consider the line **f**(L) and the point **f**(x). As x is not on L so **f**(x) is not on **f**(L).

The line **f**(L) may be considered the common edge of two half-planes H_3 and H_4.
Let the half-plane H_3 contain the point **f**(x).

We must prove that the image under **f** of each point of H_1 also lies on H_3 and equally, that each point of H_3 is mapped by $\mathbf{f^{-1}}$ to a point on H_1 .

Let $p \varepsilon H_1$, and we wish to show that $\mathbf{f}(p) \varepsilon H_3$.

Assume the contrary, that $\mathbf{f}(p) \varepsilon H_4$ but not on the edge L, as the edge is common to H_3 and H_4.



Thus $\mathbf{f}(x)$ and $\mathbf{f}(p)$ are on different sides of $\mathbf{f}(L)$, so a line segment joining the two points must cross $\mathbf{f}(L)$ at some point q.

Now $\mathbf{f}^{-1}$ maps
(i) $\mathbf{f}(L)$ to L
(ii) the line segment $[\mathbf{f}(p)\mathbf{f}(x)]$ to the line segment [px]

so $\mathbf{f}^{-1}$ maps q to the intersection of L and the line segment [px]. Now x is in H_1 but not on L, and p is not on L as $\mathbf{f}(p)$ is not on $\mathbf{f}(L)$, so the only way this intersection can contain a point is if p does not belong to H_1 - a contradiction.

So if p belongs to H_1 then $\mathbf{f}(p)$ belongs to H_3.

Conversely, let $u \varepsilon H_3$ and we now wish to show that $\mathbf{f}^{-1}(u) \varepsilon H_1$.

Both u and $\mathbf{f}(x)$ belong to H_3.

Using what we have just proved above, we can say that u and $\mathbf{f}(x)$ are both mapped by $\mathbf{f}^{-1}$ to the same half-plane. Now $\mathbf{f}(x)$ is mapped by $\mathbf{f}^{-1}$ to x, which belongs to H_1 , so u is also mapped to a point in H_1.

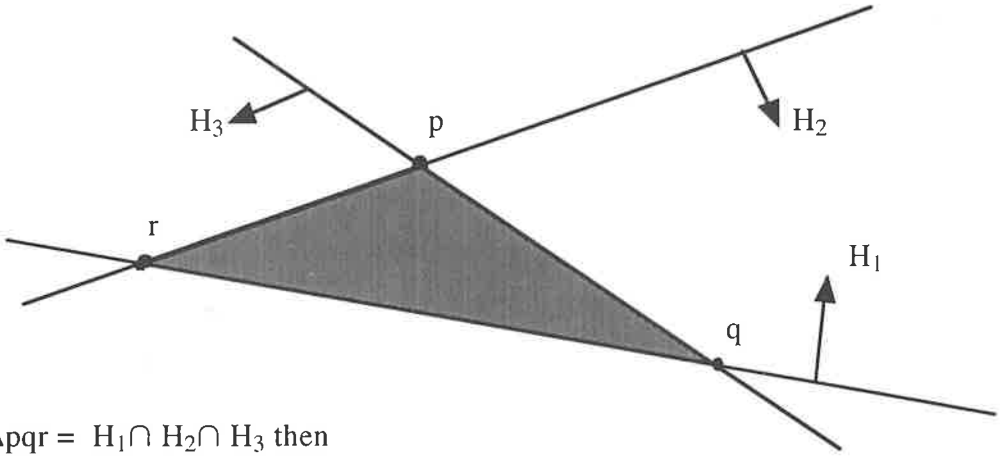
So if u belongs to H_3 then $\mathbf{f}^{-1}(u)$ belongs to H_1.

Thus $\mathbf{f}(H_1) = H_3$. **QED.**

Proof 1.5

If p,q and r are three distinct, non-collinear points of the plane, then the triangle pqr, written Δpqr, is the set of points $H_1 \cap H_2 \cap H_3$, where

H_1 is the half-plane with edge qr, containing the point p
H_2 is the half-plane with edge pr, containing the point q
H_3 is the half-plane with edge pq, containing the point r



If $\Delta pqr = H_1 \cap H_2 \cap H_3$ then

$\mathbf{f}(\Delta pqr) = \mathbf{f}(H_1 \cap H_2 \cap H_3) = \mathbf{f}(H_1) \cap \mathbf{f}(H_2) \cap \mathbf{f}(H_3)$

which is a triangle whose vertices are $\mathbf{f}$(p), $\mathbf{f}$(q), and $\mathbf{f}$(r) as required. **QED.**

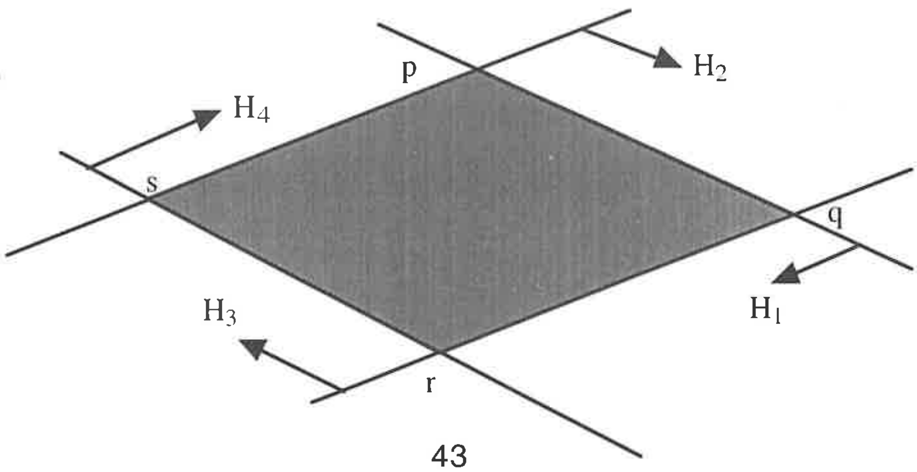
Theorem 1.6

Let pq be parallel to sr and ps be parallel to qr.

Let H_1, H_2, H_3, and H_4 be half-planes such that

H_1 has edge pq, containing the point s
H_2 has edge ps, containing the point r
H_3 has edge qr, containing the point p
H_4 has edge rs, containing the point q.

Then the parallelogram pqrs is the set of points $H_1 \cap H_2 \cap H_3 \cap H_4$.



The line **f**(p)**f**(q) will be parallel to the line **f**(s)**f**(r) as pq || rs,

and the line **f**(p)**f**(s) will be parallel to the line **f**(q)**f**(r) as ps || qr .

Moreover,
f(parallelogram pqrs) = **f**($H_1 \cap H_2 \cap H_3 \cap H_4$) = **f**(H_1)∩**f**(H_2)∩**f**(H_3)∩**f**(H_4) ,
a parallelogram whose vertices are **f**(p), **f**(q), **f**(r), and **f**(s) as required. **QED.**

We next establish some useful results concerning the area of a triangle and a very interesting interpretation of the determinant of a matrix representing an affine transformation.

Recall the following results:

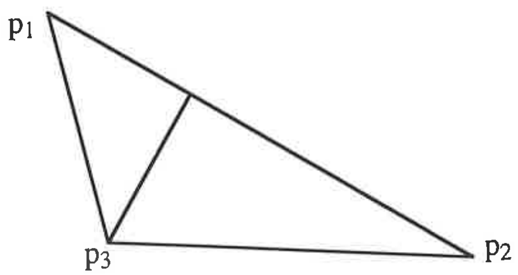
(i) Consider the points $p_1 \equiv (x_1, y_1)$ and $p_2 \equiv (x_2, y_2)$.
The equation of the line p_1p_2 is obtained from

$$\frac{(y - y_1)}{(y_2 - y_1)} = \frac{(x - x_1)}{(x_2 - x_1)} \quad \text{or}$$

$$(x_2 - x_1)(y - y_1) - (y_2 - y_1)(x - x_1) = 0.$$

(ii) The perpendicular distance of a point $p_3 \equiv (x_3, y_3)$ from a line $ax + by + c = 0$ is obtained by substituting x_3 for x and y_3 for y in the formula : distance h = $\dfrac{|ax + by + c|}{\sqrt{a^2 + b^2}}$.

(iii) The area of a triangle $= \frac{1}{2}$(length of one side)(perpendicular height)



Let p_1, p_2 , and p_3 be three non-collinear points.Then the length of the perpendicular distance h from p_3 to p_1p_2 is obtained using (i) and (ii) :

$$h = \frac{|(x_2 - x_1)(y_3 - y_1) - (y_2 - y_1)(x_3 - x_1)|}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}}$$

and the length of $[p_1p_2]$ is $= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

So the area of $\Delta p_1p_2p_3$ is obtained from (iii) as

$$\frac{1}{2}\left|(x_2 - x_1)(y_3 - y_1) - (y_2 - y_1)(x_3 - x_1)\right|$$

which becomes a more familiar result if p_3 is taken to be the origin (0,0), in which case **the area of $\Delta p_1p_2p_3$ is**

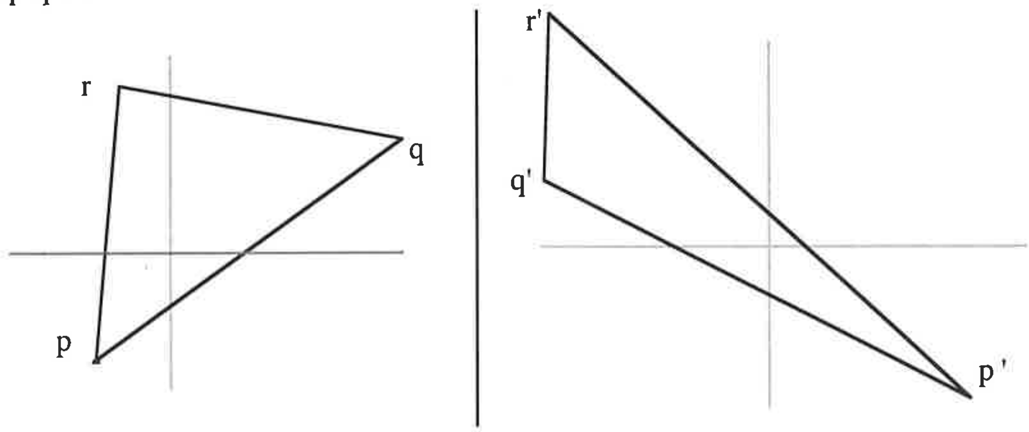
$$\frac{1}{2}\left|x_1y_2 - x_2y_1\right|$$

The modulus sign arises from the use of the formula used to calculate the perpendicular distance of a point from a line:

$$\text{distance h} = \frac{|ax + by + c|}{\sqrt{a^2 + b^2}}$$

It will be recalled that if two points are positioned on opposite sides of a line, then the value of the expression ax+by+c will change sign as the coordinates of each point are substituted into the expression. This test is often used to quickly determine if two points are on the same or opposite sides of a line.

Observe the following interesting result. Consider a triangle Δpqr and its image $\Delta p'q'r'$.



In the original triangle Δpqr, the vertices are traversed anti-clockwise in the sequence p,q,r. In the image triangle, in this example, the vertices are traversed anti-clockwise in the sequence p',r',q'. This **change in orientation** may be detected very quickly by finding the area of each triangle, for the value of the expression inside the modulus sign will have an opposite sign as each area is calculated.

We next establish a relationship between the area of a triangle and the area of its image triangle under an affine transformation.

Consider a triangle $\Delta p_1p_2p_3$ with $p_1 \equiv (x_1,y_1)$, $p_2 \equiv (x_2,y_2)$ and $p_3 \equiv (0,0)$ and

$\text{area}(\Delta p_1p_2p_3) = \frac{1}{2}\left|x_1y_2 - x_2y_1\right|$.

We apply an affine transformation of type

$x' = ax + by + 0$

$y' = cx + dy + 0$ with $ad - bc \neq 0$

or, in matrix form

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \text{ with } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } ad - bc \neq 0.$$

Note that the determinant of A : $\det(A) = (ad - bc)$.

Appying this transformation to the triangle $\Delta p_1 p_2 p_3$

we obtain an image triangle with vertices

$p_1' \equiv (ax_1 + by_1, cx_1 + dy_1)$

$p_2' \equiv (ax_2 + by_2, cx_2 + dy_2)$

$p_3' \equiv (0,0)$

Applying the area formula to these co-ordinates gives

the area of the image triangle $\Delta p_1' p_2' p_3' =$

$$\frac{1}{2}\left|(ax_1 + by_1)(cx_2 + dy_2) - (ax_2 + by_2)(cx_1 + dy_1)\right|$$

which simplifies to

$$\frac{1}{2}\left|(ad - bc)(x_1 y_2) - (ad - bc)(x_2 y_1)\right|$$

$$= \frac{1}{2}|ad - bc||x_1 y_2 - x_2 y_1|$$

$$= |\det A| \cdot (\text{area of } \Delta p_1 p_2 p_3)$$

that is,

$$\text{area of } \Delta p_1' p_2' p_3' = |\det A| \cdot (\text{area of } \Delta p_1 p_2 p_3)$$

Thus we may interpret the modulus of the determinant of the matrix of an affine transformation as a measure of the effect of the transformation upon the area of a triangle to which the transformation is applied.

The work shown here establishes this result for the restricted case of a triangle with one vertex at (0,0) under an affine transformation with k_1 and k_2 both zero. With a little more algebra, and using the more general formula for the area of a triangle, the general result stated here may be demonstrated to be true.

An immediate consequence of this result is that as an affine transformation maps a triangle to a triangle, it will only conserve the area of the triangle if the determinant of its matrix has modulus 1. This matter will be dealt with in more depth below, but a class exercise might be to express this condition in terms of the entries in the matrix and to provide some examples of affine transformations under which area of triangles is an invariant. Further, can this result be extended to other plane figures such as a parallelogram? A square? A polygon? A circle?

We now show some properties that are not, in general, invariant by exhibiting one affine transformation which does not conserve them.

The Power of Counter Examples

Let **f** be the transformation defined by p'= **f**(p) , where p is(x,y) and p' is (x',y') , and for all (x,y) ,

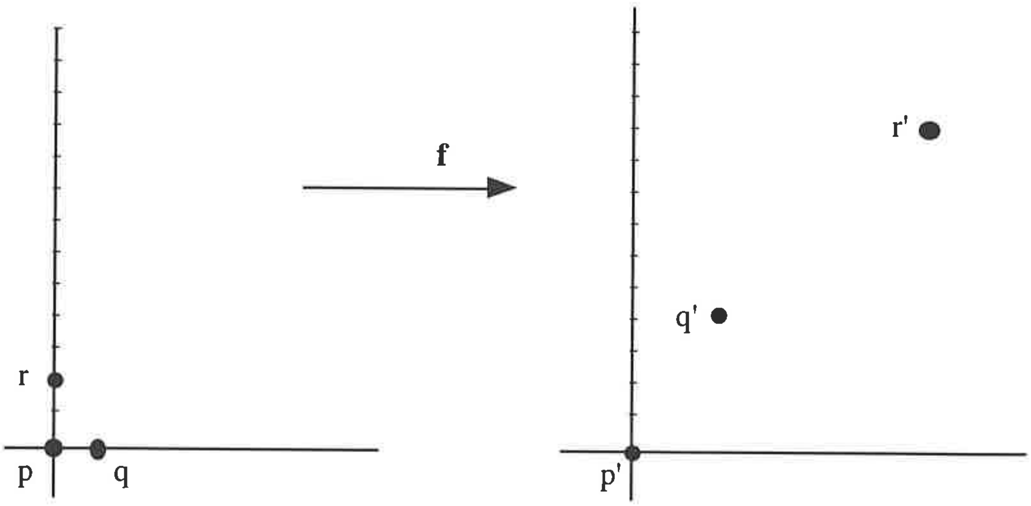
$$x' = 2x + 3y$$
$$y' = 4x + 5y$$

In matrix form this looks like

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ , with } \det\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix} = -2 \neq 0$$

so **f** is an affine transformation.

Let us now take the points p,q and r with coordinates (0,0), (1,0) and (0,2) respectively. Then the image points p', q', and r' will have coordinates (0,0), (2,4) and (6,10) respectively.



Using the distance formula

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

gives | pq | = 1 **but** | p'q' | = √20 so that | pq | ≠ | p'q' | .

Thus not all affine transformations preserve all distances , and so

distance is not an affine invariant

We also note that | pr | = 2 and | p'r' | = √136, so

$$\frac{|pr|}{|pq|} = 2 \text{ , } \frac{|p'r'|}{|p'q'|} = \sqrt{\frac{136}{20}} \text{ .}$$

Thus

$$\frac{|pr|}{|pq|} \neq \frac{|p'r'|}{|p'q'|} \text{ .}$$

Thus not all affine transformations preserve all ratios of distances, and so

the ratio of two distances is not an affine invariant

We note now that the line pq is perpendicular to the line pr.

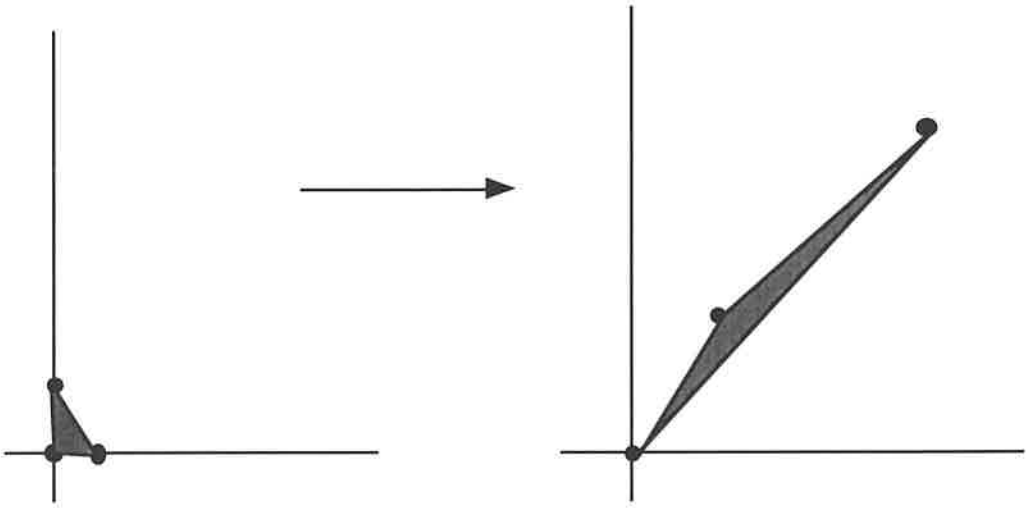
Using the slope formula $\frac{y_2 - y_1}{x_2 - x_1}$

we obtain the slope of p'q' to be 2 and the slope of p'r' to be 5/3. That is, the image lines are not perpendicular to eachother, as the product of their slopes is not -1.

Thus not all affine transformations preserve right-angles, so

neither perpendicularity nor angle-size is an affine invariant

Further, we note that the triangle Δpqr is mapped to the triangle Δp'q'r'.



Applying the formula for the area of a triangle gives the area of Δpqr to be 1 and the area of Δp'q'r' to be 2, so the areas of these two triangles are not equal. Thus not all affine transformations preserve the areas of all triangles, and so

area of a triangle is not an affine invariant

Finally we note that in the Δpqr the vertices are traversed anti-clockwise in the sequence p,q,r. In the image triangle Δp'q'r' however, the sequence p', q', r' is traversed clockwise. (Note that the determinant of the transformation is -2 , ie **negative**). Thus not every affine transformation preserves the orientation of three non-collinear points as either anti-clockwise or clockwise, and so

orientation is not an affine invariant

Although we have seen above that , in general, distance is not an affine invariant and that the ratio of two distances is not, in general, an affine invariant, it will turn out to be very productive if we obtain an expression for the ratio of the length of an image line segment to the length of the original line segment. This ratio is known as the **magnification ratio**. This approach leads to a new result:

If **f** ε *Cl*, then the ratios of lengths of segments on parallel lines are invariant under **f** **1.7**

Behaviour of distance under affine transformations.

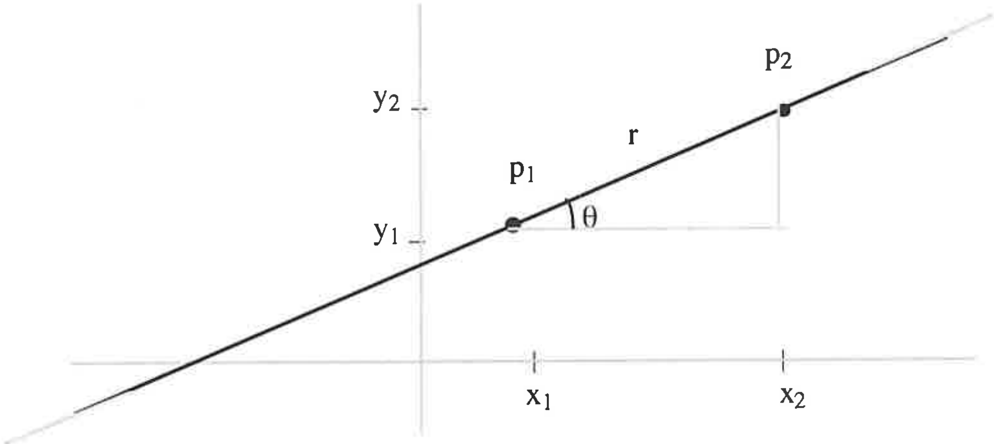
Let **f** be an affine transformation of the plane $\prod$ which has the co-ordinate form

$$(x,y) \rightarrow (x',y) \quad \text{where}$$

$$x' = ax + by + k_1$$
$$y' = cx + dy + k_2$$

$$\text{and} \quad ad - bc \neq 0 \ ,$$

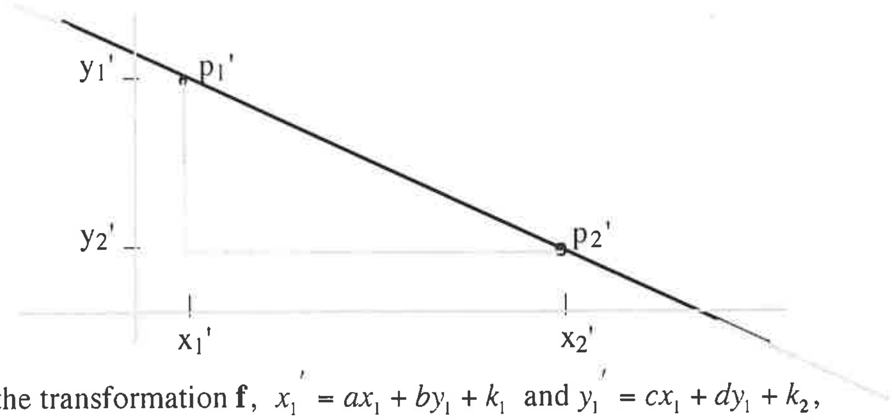
and let p_1 and p_2 be distinct points with coordinates (x_1,y_1) and (x_2,y_2) respectively. Let $|\ p_1p_2| = r$ and let the half-line $[p_1p_2$ have angle of inclination θ .



Then,

$x_2 - x_1 = r\cos\theta\,, \quad y_2 - y_1 = r\sin\theta.$

We now apply an affine transformation **f** to the points p_1 and p_2 and obtain the image points p_1' and p_2'. We wish to calculate the distance $|\ p_1'p_2'\,|$.



Under the transformation **f**, $x_1' = ax_1 + by_1 + k_1$ and $y_1' = cx_1 + dy_1 + k_2$,

$x_2' = ax_2 + by_2 + k_1$ and $y_2' = cx_2 + dy_2 + k_2$.

So $x_2' - x_1' = a(x_2 - x_1) + b(y_2 - y_1) = ar\cos\theta + br\sin\theta = r(a\cos\theta + b\sin\theta)$.

Similarly,

$$y_2' - y_1' = r(c\cos\theta + d\sin\theta).$$

Then the ratio of

$$\frac{\text{length of the line segment from } p_1' \text{ to } p_2'}{\text{length of the line segment from } p_1 \text{ to } p_2} = \frac{\sqrt{\left(x_2' - x_1'\right)^2 + \left(y_2' - y_1'\right)^2}}{r}$$

$$= \frac{\sqrt{r^2(a\cos\theta + b\sin\theta)^2 + r^2(c\cos\theta + d\sin\theta)^2}}{r}$$

$$= \frac{r\sqrt{(a\cos\theta + b\sin\theta)^2 + (c\cos\theta + d\sin\theta)^2}}{r}$$

$$= \sqrt{(a\cos\theta + b\sin\theta)^2 + (c\cos\theta + d\sin\theta)^2}$$

We refer to this ratio as the **magnification ratio** k.

If we refer back to the example on page 47, this ratio k should be √20.
In that example a=2, b=3, c=4, d=5, θ=0.
Using the expression for k we have just obtained shows k to be

$$\sqrt{(2\cos 0 + 3\sin 0)^2 + (4\cos 0 + 5\sin 0)^2} = \sqrt{20}$$

However, the really important information we can extract from this complicated expression for k is simple to see - if you stand back far enough!

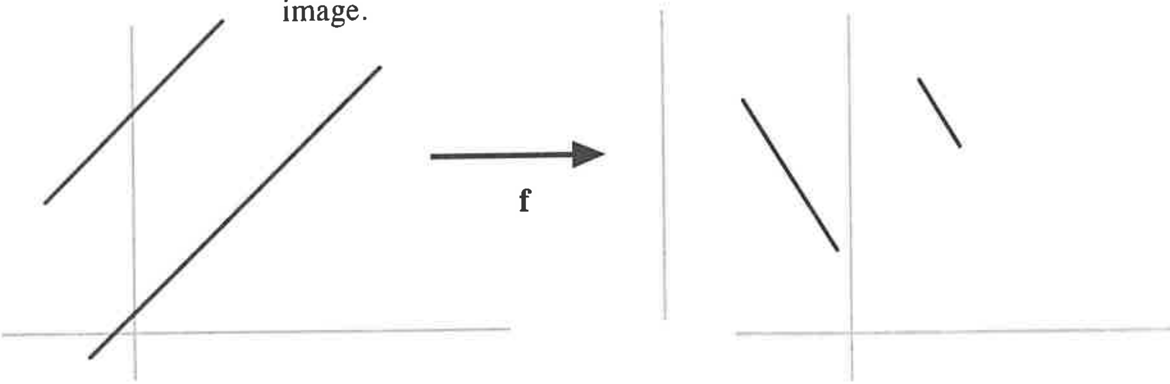
Proof 1.7

The magnification ratio k varies with **f** (because of a,b,c, and d) and with θ.

We conclude from this:

(i) if we apply different transformations to the same line segment the resulting value of k is likely to be different because k varies with **f**

(ii) if we apply the same transformation (ie the values of a, b, c and d remain the same) to two line segments **which are on parallel lines** (ie which have the same value of θ, or more exactly, θ ± π), then **the value of k must remain the same** for each line segment and its image.



that is,

$$\frac{|p_1'p_2'|}{|p_1p_2|} = \frac{|p_3'p_4'|}{|p_3p_4|}.$$

Rearranging this gives

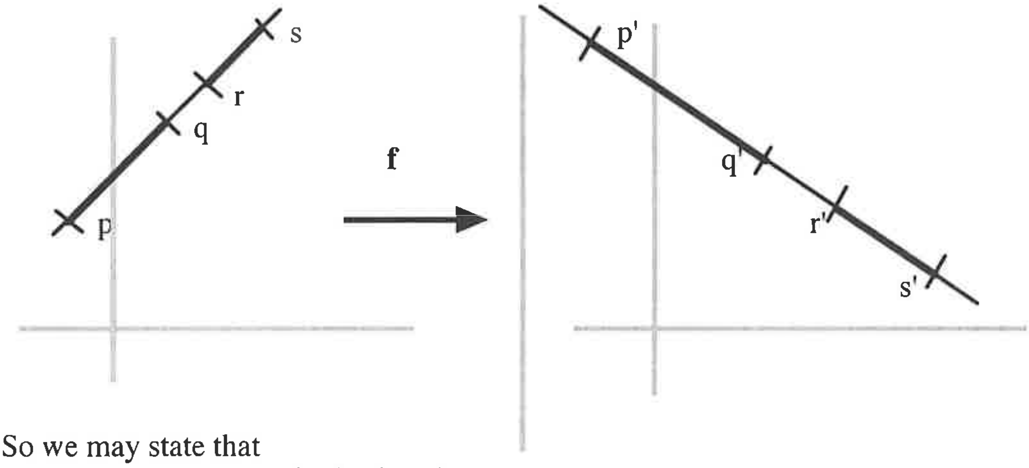
$$\frac{|p_3p_4|}{|p_1p_2|} = \frac{|p_3'p_4'|}{|p_1'p_2'|}.$$

Thus, ratios of lengths of segments on parallel lines are invariant under affine transformation. **QED.**

This is a very important result.

We saw above that the ratio of two distances is not an affine invariant.
We have now proved something about a more restricted case, in which the distances are the lengths of line segments on **parallel** lines.

It will be recalled that a line is parallel to itself.



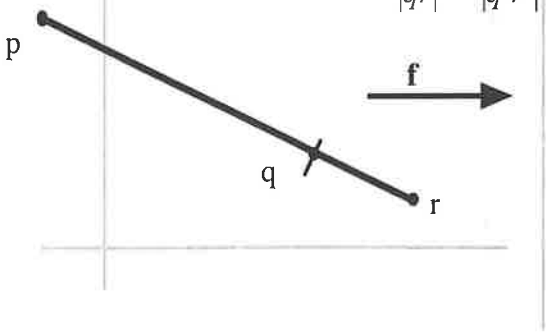
So we may state that

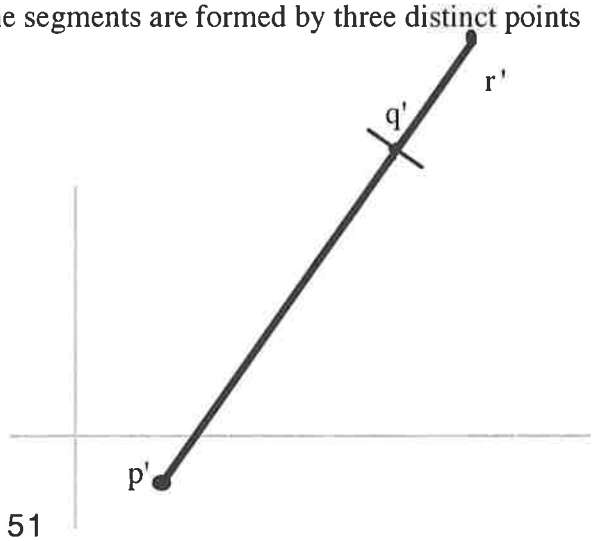
$$\frac{|pq|}{|rs|} = \frac{|p'q'|}{|r's'|}$$

A very productive case is when the line segments are formed by three distinct points p,q and r.

Thus

$$\frac{|pq|}{|qr|} = \frac{|p'q'|}{|q'r'|}$$





We now obtain the following results:

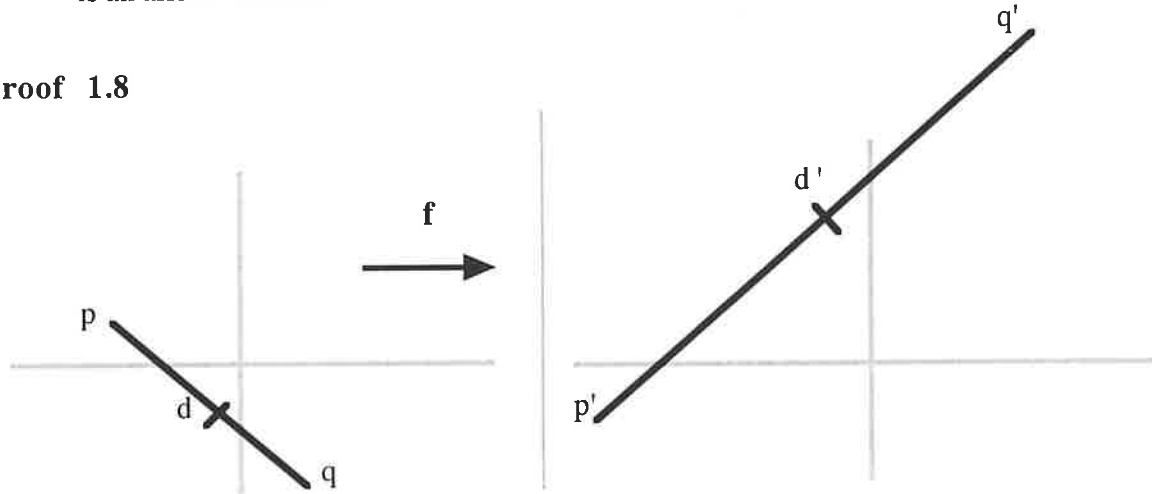
if $\mathbf{f} \, \varepsilon \, Cl$, then

being a midpoint is an affine invariant **1.8**

being the centroid of a triangle is an affine invariant **1.9**

internal and external division of a line segment in equal ratios is an affine invariant **1.10**

Proof 1.8



Let p and q be distinct points with d the midpoint of $[pq]$.

Then the three points are collinear, and $|pd| = |dq|$

or, $\dfrac{|pd|}{|dq|} = 1$ and $pd \parallel dq$.

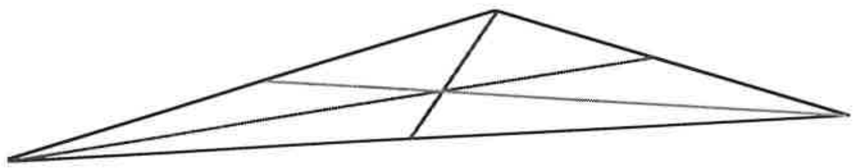
If $\mathbf{f}$ is an affine transformation, then

$\mathbf{f}(p) = p'$, $\mathbf{f}(q) = q'$ and $\mathbf{f}(d) = d'$ so

p', q' and d' are collinear, and by **1.7**, $\dfrac{|p'd'|}{|d'q'|} = 1$.

Hence d' is the midpoint of $[p'q']$. **QED**

Proof 1.9



Recall that a **median** is a line joining a vertex of a triangle and the midpoint of the side opposite the vertex.
Recall that the medians of a triangle are concurrent in a point called the **centroid**.

Let $\mathbf{f}$ be an affine transformation.
Then $\mathbf{f}$ maps a triangle to a triangle (vertices to vertices, sides to sides).
By **1.8**, midpoints of sides are mapped to midpoints of sides.
Consequently, each median is mapped to a median, so the centroid of a triangle is mapped to the centroid of the image triangle. **QED**

Proof 1.10

Consider the line segment [pq] on the line pq. Let r and s be two points of pq such that r and s divide [pq] internally and externally in the same ratio.
That is,

$$\frac{|pr|}{|rq|} = \frac{|ps|}{|sq|}$$

and all segments are parts of the one line,which is parallel to itself.

If **f** is an affine transformation, then

$\mathbf{f}(p) = p'$, $\mathbf{f}(q) = q'$ and $\mathbf{f}(r) = r'$ and $\mathbf{f}(s) = s'$.

So by **1.7**,

$$\frac{|p'r'|}{|r'q'|} = \frac{|p's'|}{|s'q'|}$$

that is, internal and external division of a segment in equal ratios is an affine invariant. **QED**

The final result in this section considers the ratio of the areas of two triangles and what happens to this ratio under an affine transformation. This ratio proves to be invariant, so consequently

the ratio of the areas of two triangles is an affine invariant **1.11**

Proof 1.11

Recall the result shown on page 46

$$\text{area of } \Delta p_1' p_2' p_3' = |\det A| \,.\, (\text{area of } \Delta p_1 p_2 p_3)$$

where the triangle $\Delta p_1 p_2 p_3$ is mapped by an affine transformation **f** to

the image triangle $\Delta p_1' p_2' p_3'$, where **f** is expressed in the form

$$A\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}.$$

If $\Delta p_1 p_2 p_3$ and $\Delta p_4 p_5 p_6$ are two triangles then

$$\text{area } \Delta p_1' p_2' p_3' = |\det A| \,.\, (\text{area } \Delta p_1 p_2 p_3) \text{ and}$$

$$\text{area } \Delta p_4' p_5' p_6' = |\det A| \,.\, (\text{area } \Delta p_4 p_5 p_6).$$

Thus $$\frac{\text{area } \Delta p_1' p_2' p_3'}{\text{area } \Delta p_4' p_5' p_6'} = \frac{|\det A| \,.\, (\text{area } \Delta p_1 p_2 p_3)}{|\det A| \,.\, (\text{area } \Delta p_4 p_5 p_6)} = \frac{\text{area } \Delta p_1 p_2 p_3}{\text{area } \Delta p_4 p_5 p_6}.$$

Hence the ratio of the areas of two triangles is an affine invariant.

QED

At this point it is important that we do something with our results. As already indicated, the area of investigation will be the transformation which maps a circle, centre (0,0), radius 1 to an ellipse whose equation is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

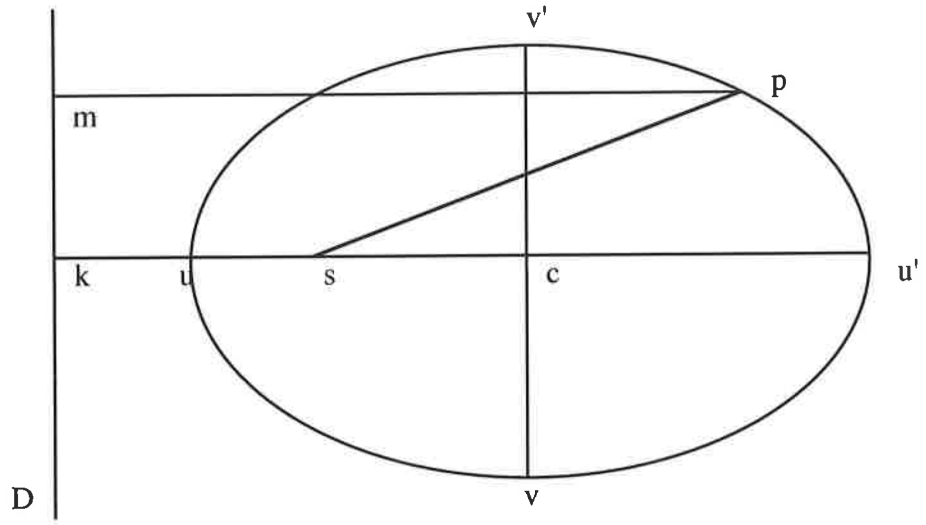
and the deduction from results for a circle of similar results for an ellipse.

As the syllabus refers to

> Focus-directrix definition of an ellipse; derivation of the equation of an ellipse in standard form

we had better provide these before proceeding further.

2 : Focus-directrix definition of an ellipse



Let D be a fixed line, s $\notin$ D a fixed point, and e > 0 a fixed number.

Then the set C ={ p: |ps| = e|pm| , where m is the foot of the perpendicular from p to D} is called a *conic section* . We call s a *focus* , D a *directrix* and e the *eccentricity* of C.

When

(i) $e = 1$, we call C a *parabola* P ;

(ii) $0 < e < 1$, we call C an *ellipse* E ;

(iii) $e > 1$, we call C a *hyperbola* H .

Equation of an ellipse in standard form

Proof 2.1

Let k be the foot of the perpendicular from s to D. We divide [ks] internally and externally in the ratio *1:e* , $0 < e < 1$, and thus take points u and u' such that u $\in$ [ks] , s $\in$ [ku'] and

|su| = e|uk| , |su'| = e|u'k| .

Then u and u' are points of the ellipse E on the line sk. We let c be the mid-point of [uu'] and let |cu| = |cu'| = a. We choose our coordinate system so that sk is the X-axis, and c is the origin, so that u and u' have coordinates (-a,0) and (a,0) respectively.

Then s has coordinates (-ae,0) and k has coordinates (-a/e,0).

The directrix D has the equation ex + a = 0.

As E ={ p: |ps| = e|pm| , where |pm| is the perpendicular distance from p to D},

then p(x,y) is on the ellipse E if and only if

$$\sqrt{(x+ae)^2+(y-0)^2} = e\left(\frac{|ex+a|}{\sqrt{e^2+0^2}}\right).$$

As each side is ≥0, we may square each side to obtain

$$(x+ae)^2+y^2=(ex+a)^2$$

which gives

$$(1-e^2)x^2+y^2=a^2(1-e^2)$$

and hence

$$\frac{x^2}{a^2}+\frac{y^2}{a^2(1-e^2)}=1.$$

Letting $a^2(1-e^2)=b^2$, where $b>0$, we obtain

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1. \qquad 2.1.1$$

QED.

[uu'] is called the *major axis*, [vv'] is called the *minor axis*, where v has coordinates (0,-b) and v' has coordinates (0,b) .

If p(x,y) is on the ellipse, satisfying equation 2.1.1, so do the points (x,-y) and (-x,y). Thus E is symmetrical about the X-axis and symmetrical about the Y-axis.

3 : Circle and ellipse

As we have already seen, the transformation **f** given by the equations

$$x' = ax$$
$$y' = by$$

or in matrix form

$$\begin{pmatrix}x'\\y'\end{pmatrix}=\begin{pmatrix}a&0\\0&b\end{pmatrix}+\begin{pmatrix}0\\0\end{pmatrix}$$

maps the circle $x^2+y^2=1$ to the ellipse $\frac{x'^2}{a^2}+\frac{y'^2}{b^2}=1$, with $a>b>0$.

As $\det\begin{pmatrix}a&0\\0&b\end{pmatrix}=ab$, which is non - zero, **f** is an affine transformation.

Consequently, **f** must have an inverse, which is also affine. This inverse, $\mathbf{f}^{-1}$, is given by

$$x'=\frac{1}{a}x \ , \ y'=\frac{1}{b}y .$$

Following from **1.1**, **1.2**, and **1.3**, the transformations **f** and $\mathbf{f}^{-1}$ map each line to a line, each line segment to a line segment, and each pair of parallel lines to a pair of parallel lines.

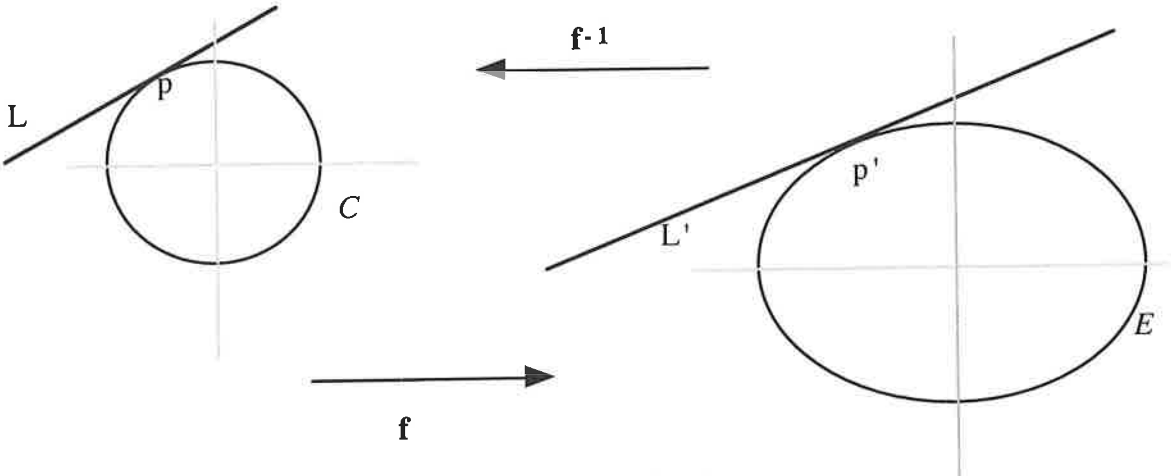
We now apply these ideas to deduce some results for an ellipse from results we already know for a circle. In what follows, points or lines which are dashed are connected with an ellipse, otherwise points or lines are connected with a circle.

Let L' be any line. Then $\mathbf{f}^{-1}$(L') is a line. Let $\mathbf{f}^{-1}$(L') = L.

The line L will meet the circle *C* in exactly two points **or** exactly one point **or** in no point at all.

Consequently L' will meet the ellipse *E* in exactly two points **or** exactly one point **or** in no point at all.

When L meets *C* in one point p, L is the tangent to *C* at p.
In that case L' meets *E* in one point p', and then L' is the tangent to *E* at p'.



Note that the origin o = (0,0) is the centre of *C* and that **f**(o) = o.

Deduction 3.1

The centre of an ellipse *E* is the midpoint of every chord which contains it.

Proof 3.1

Consider any distinct points p_1' and p_2' which belong to an ellipse *E*.
The segment $[p_1'p_2']$ is called a *chord* of the ellipse *E*.
The line $p_1'p_2'$ meets *E* only in the points p_1' and p_2' .
Suppose that o lies on the chord $[p_1'p_2']$.
Then o = $\mathbf{f}^{-1}$(o) lies on the chord $[p_1p_2]$ of the circle *C* .
But o is the midpoint of $[p_1p_2]$, which must be diameter.
By **1.8** , **f** maps midpoints of segments to midpoints of segments, so o = **f**(o) is the midpoint of $[p_1'p_2']$.

QED

The point o is called the **centre** of E, and every chord of E containing o is called a **diameter** of E.

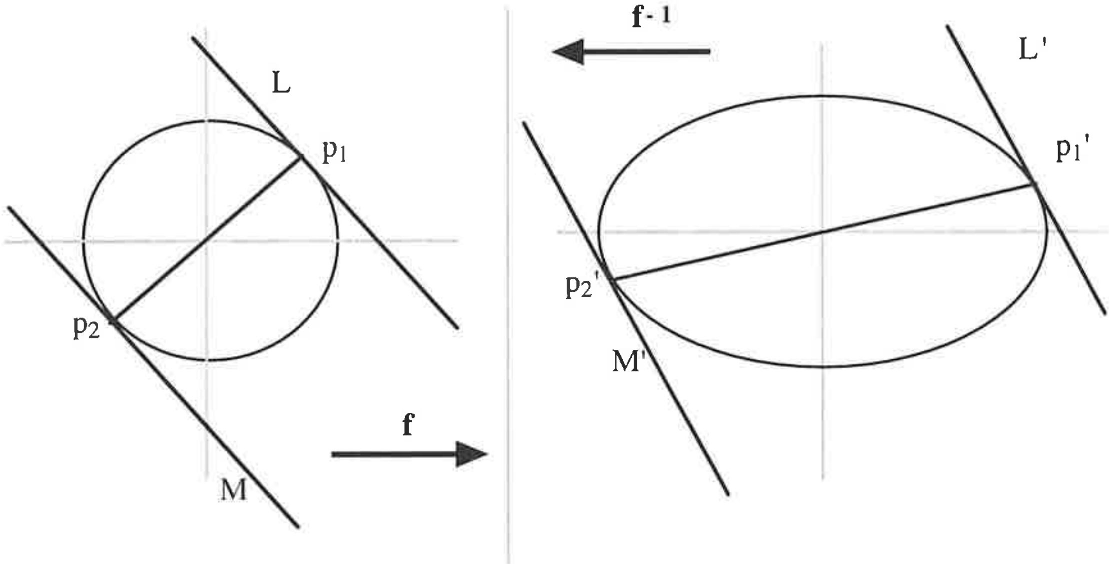
Note: the point o is unique, for if o and o_1 are distinct points, then oo_1 can meet E only in two points p' and q', and both o and o_1 cannot both be the midpoint of [p'q'].

We now prove the result we met at the very beginning:

Deduction 3.2

Tangents to an ellipse at the end-points of a diameter are parallel to each other.

Proof 3.2



Let $[p_1'p_2']$ be any diameter of the ellipse E, and let L' and M' be the tangents to E at p_1' and p_2' respectively.

Then $L = \mathbf{f}^{-1}(L')$ and $M = \mathbf{f}^{-1}(M')$ are tangents to the circle C at the points p_1 and p_2.

As $[p_1'p_2']$ is a diameter of E it contains the origin o.

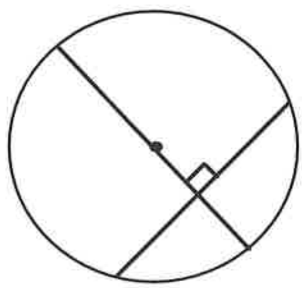
So $[p_1p_2]$ contains the image of the origin o under $\mathbf{f}^{-1}$ and $\mathbf{f}^{-1}(o)=o$.

Thus $[p_1p_2]$ is a diameter of C, so the tangents L and M are parallel to each other.

As **f** maps parallel lines to parallel lines, so L' || M'. **QED**

The next deduction calls upon a property of the diameter of a circle.

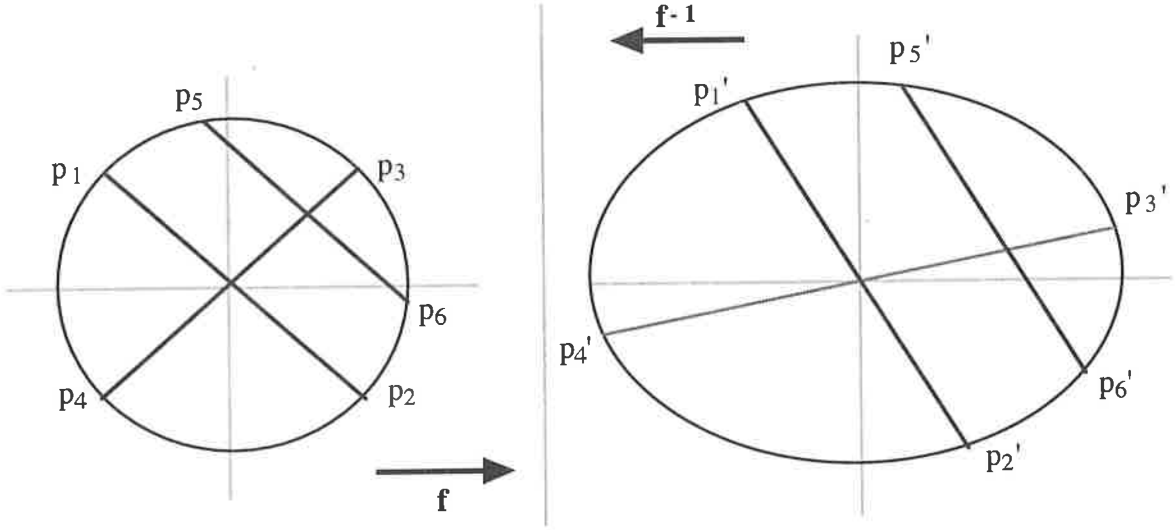
If a diameter of a circle is perpendicular to a chord, then the diameter bisects the chord. That is, the diameter contains the midpoint of every chord perpendicular to it.



Deduction 3.3

The locus of midpoints of parallel chords of an ellipse E is a diameter (less its end-points) of E.

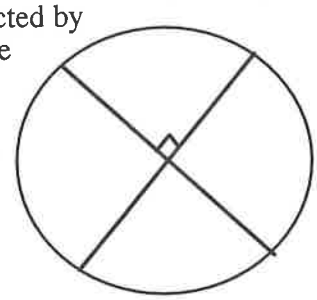
Proof 3.3



Let $[p_1'p_2']$ be any diameter of the ellipse E.
Consider the chords of type $[p_5'p_6']$, where $p_5'p_6'$ is parallel to $p_1'p_2'$.
Under the affine transformation $\mathbf{f^{-1}}$, $[p_1'p_2']$ is mapped to the circle diameter $[p_1p_2]$, and chords of type $[p_5'p_6']$ are mapped to chords of type $[p_5p_6]$, all on lines parallel to p_1p_2 .
Let $[p_3p_4]$ be a diameter of the circle C, such that p_3p_4 is perpendicular to p_1p_2 and also perpendicular to all lines parallel to p_1p_2.
Then $[p_3p_4]$ bisects all the chords of type $[p_5p_6]$, and so contains their midpoints. In fact, such midpoints fill out all of $[p_3p_4]$, except for its end-points.
As **f** maps midpoints to midpoints, the midpoints of all chords $[p_5p_6]$ are mapped to midpoints of the chords $[p_5'p_6']$ and these points in turn fill out the diameter $[p_3'p_4']$ except for its end-points. **QED**

The next deduction requires an new idea, that of the **conjugate diameters of an ellipse**. We have already recalled a property of the diameter of a circle, that a diameter bisects all chords which are perpendicular to it.

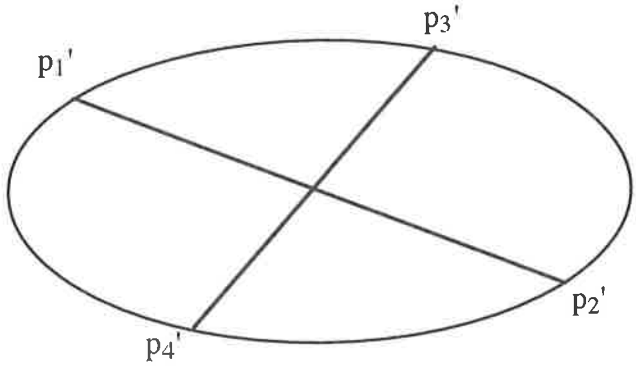
When two diameters of a circle are perpendicular to each other, as shown here, then all chords which lie on lines parallel to one diameter are bisected by the other diameter - **and vice versa**. Such a pair of diameters are known as the **conjugate diameters** of a circle.



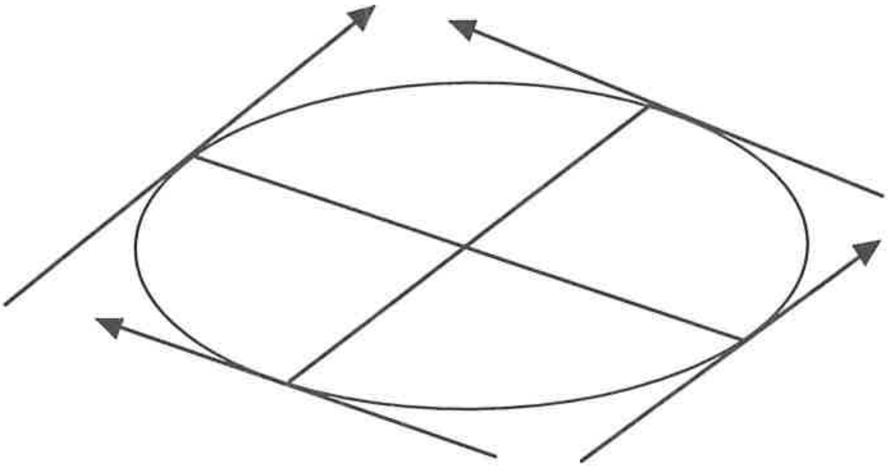
If we deal with bisectors as lines containing midpoints, as we did in **3.3** above, we may map this idea of conjugate diameters over to an ellipse.

Thus we may say that given a diameter $[p_1'p_2']$ of an ellipse E, there is a second diameter $[p_3'p_4']$, such that $[p_1'p_2']$ bisects all chords of E on lines parallel to $[p_3'p_4']$ and vice versa.

Such diameters are called **conjugate diameters of** E.

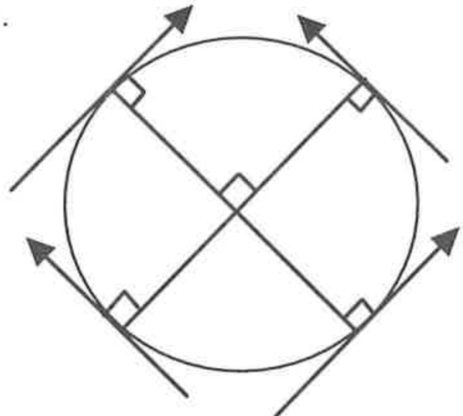


An interesting property of the conjugate diameters of an ellipse concerns the tangents at their end-points.



In this diagram we have a pair of conjugate diameters of an ellipse, with the tangents drawn in at the end-points of each diameter. We have already proved (**3.2**) that each pair of tangents is parallel.

As usual, we map this situation on the ellipse back to the more familiar ground of the circle in the hope of picking up some information which, in turn, is capable of being mapped back unchanged to the ellipse.

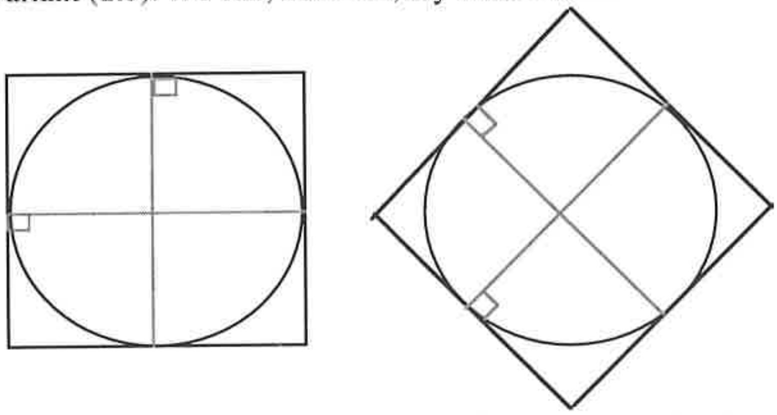


Here we see the result of such a mapping. The conjugate diameters of the ellipse have become two diameters of a circle, at right angles to each other. The tangents at the end-points of each diameter are perpendicular to the diameter and parallel to each other. Interestingly, the pair of tangents at the ends of one diameter are parallel to the other, conjugate diameter.

As we map back to the ellipse, the right angles are not preserved, as we have already noted that, in general, the measure of an angle is not an affine invariant. However, parallelism **is** an affine invariant, and consequently we may conclude that the **tangents at the end of one conjugate diameter of an ellipse are parallel to the other conjugate diameter - and vice versa**.

(Class exercise- write out a more formal proof based on this discussion).

A glance back at the diagrams on page 57 helps to see the basis of our next result. The tangents to the end-points of the conjugate diameters of the ellipse will form a parallelogram. These tangents mapped over to the circle will also form a parallelogram, as this is an affine invariant (**1.6**). We can, however, say a little more.



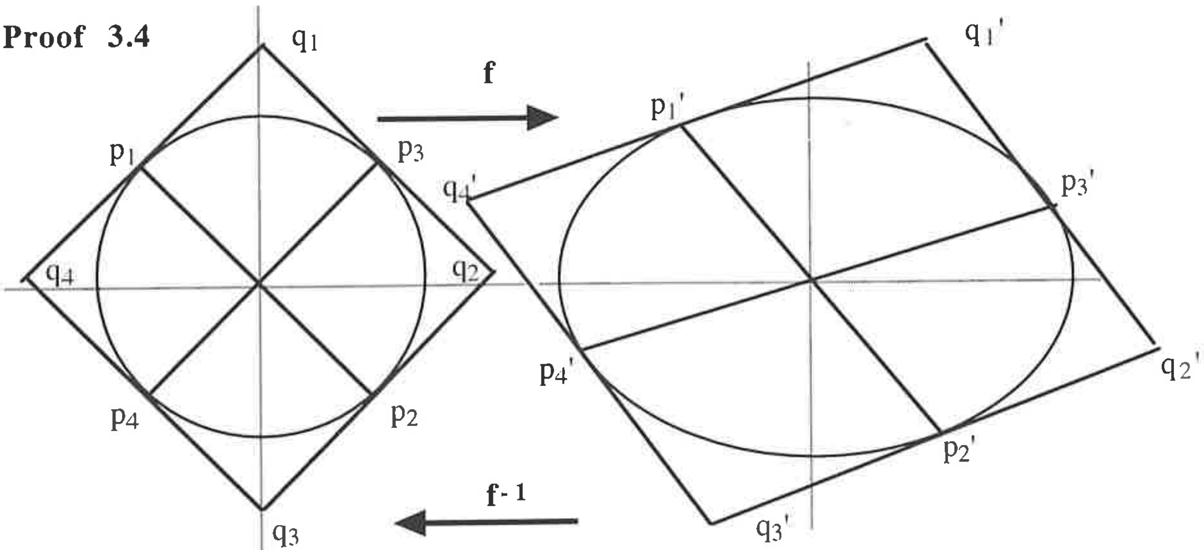
These two pictures show that the parallelogram around the circle is in fact a square - a square whose area is constant. If the radius of the circle is r, then the area of the circumscribing square is $4r^2$.

We are now in a position to deduce a result about the parallelogram around the ellipse:

Deduction 3.4

For an ellipse *E*, the parallelogram formed by the tangents at the end-points of a pair of conjugate diameters has constant area.

Proof 3.4



Let $[p_1'p_2']$ and $[p_3'p_4']$ be conjugate diameters of an ellipse *E*.
The tangents at the end-points of these diameters form a parallelogram with vertices q_1', q_2', q_3', q_4', say.

Under the affine transformation $\mathbf{f}^{-1}$, the parallelogram circumscribing the ellipse is mapped to a square circumscribing the circle.
The vertices of the square are q_1, q_2, q_3, q_4.
All such squares have the same area.
If q_1 and q_3 are opposite vertices, then the area of the square $q_1q_2q_3q_4$ is twice the area of the triangle $q_1q_2q_3$. That is, the ratio of the area of the square to the area of the triangle is **2:1**.
The affine transformation **f** will map the square $q_1q_2q_3q_4$ to the parallelogram $q_1'q_2'q_3'q_4'$ and the triangle $q_1q_2q_3$ to the triangle $q_1'q_2'q_3'$, keeping the ratio of areas intact as an affine invariant.

Thus	$\text{area}(q_1'q_2'q_3'q_4')$	=	$2 \cdot \text{area}(q_1'q_2'q_3')$
but	$\text{area}(q_1'q_2'q_3')$	=	$\lvert\det A\rvert \cdot \text{area}(q_1q_2q_3)$
hence	$\text{area}(q_1'q_2'q_3'q_4')$	=	$2 \cdot \lvert\det A\rvert \cdot \text{area}(q_1q_2q_3)$
		=	$\lvert\det A\rvert \cdot 2 \cdot \text{area}(q_1q_2q_3)$
		=	$\lvert\det A\rvert \cdot \text{area}(q_1q_2q_3q_4)$

But $\text{area}(q_1q_2q_3q_4)$ is constant, as is the value of det A.

Consequently, $\text{area}(q_1'q_2'q_3'q_4')$ is constant. **QED**

NB: $\lvert\det A\rvert = ab$ and $\text{area}(q_1q_2q_3q_4) = 4r^2 = 4$, as $r=1$.
thus $\text{area}(q_1'q_2'q_3'q_4') = ab \cdot 4 = 4ab$
a result which agrees with the case when the parallelogram circumscribing an ellipse is a rectangle with sides parallel to the X and Y axes.

The final result of this section concerns a property of the circle concerning the locus of harmonic conjugates of a point (key terms here are **pole** and **polar**).

The process of establishing a similar result for an ellipse is really straightforward, and follows the pattern of proof which hopefully is now a familiar one. The key affine invariant here is the result established at **1.10**, that internal and external division of a line segment is an affine invariant.

The real difficulty here is to establish the result for the circle, and this is done through a sequence of three results, which are sketched out below.

Note 1

Let a and b be distinct points, and m their midpoint.
Let c and d be points of the line ab, on the one side of m and satisfying

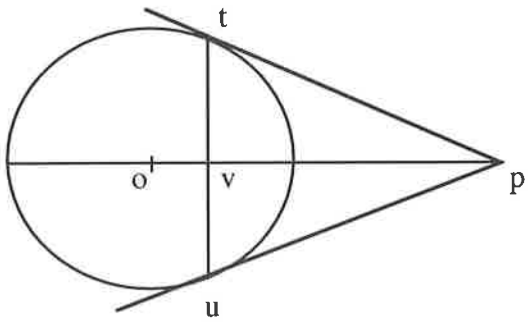
$$|mc|.|md| = |mb|^2$$

Then c and d divide the segment [ab] internally and externally in the same ratio.

Note 2

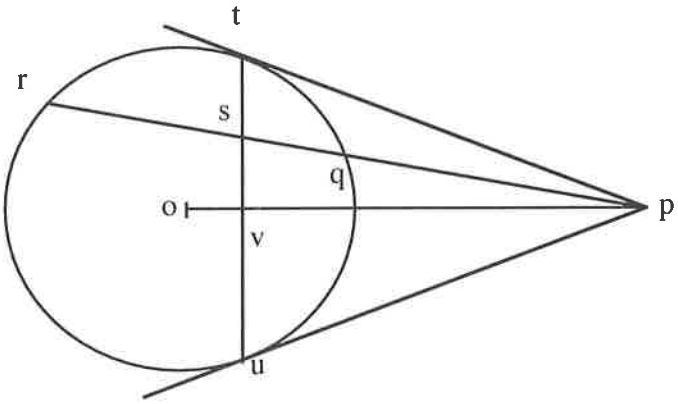
Let t and u be the points of contact of the tangents from an exterior point p to a circle *C* with centre o and length of radius k. Then the chord [tu] meets the segment [op] in a point v such that

$$|ov| \, . \, |op| = k^2$$



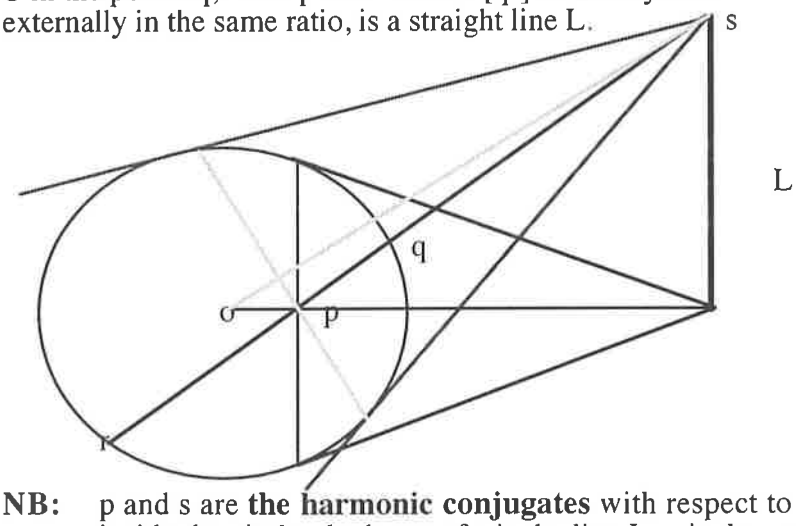
Note 3

Let t and u be the points of contact from an exterior point p to a circle *C* with centre o. If a line through p cuts *C* in the points q and r, and cuts tu in s, then s and p divide the segment [qr] internally and externally in the same ratio.



Note 4

Let *C* be a circle with centre o, and p any interior point other than o. Then the locus of points s such that if a line sp meet *C* in the points q,r then p and s divide [qr] internally and externally in the same ratio, is a straight line L.

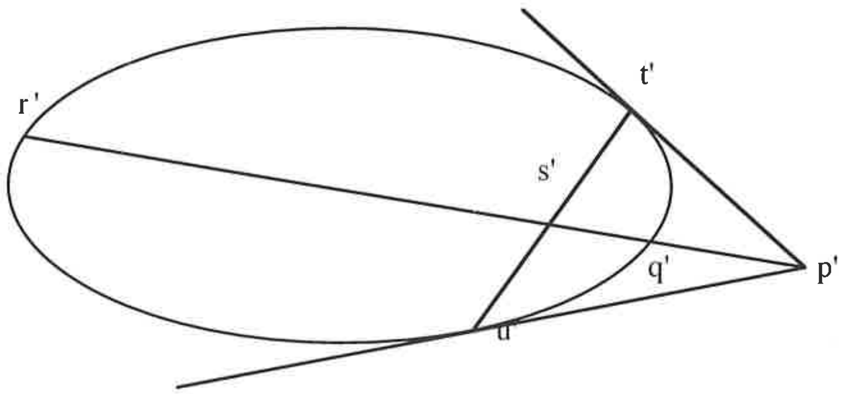


NB: p and s are **the harmonic conjugates** with respect to the circle.If p is fixed inside the circle, the locus of s is the line L. p is the **pole** and L the **polar**.

Deduction 3.5

For an ellipse *E,* centre o, and p' any interior point other than o, the locus of points s' such that if a line s'p' meet *E* in the points q',r' then p' and s' divide [q'r'] internally and externally in the same ratio, is a straight line.

If p' is an exterior point of *E*, then the locus of s' is [t'u'], where t' and u' are the points of contact of tangents from p' to *E*.



Proof 3.5

The ellipse, lines and points as given above are mapped by the affine transformation $\mathbf{f}^{-1}$ to a circle.

By **1.10**, the internal and external division of a line segment is invariant.

Consequently, the locus of s, the image of s', is a line L . (**Note 4**).

Under the transformation **f**, the locus of s, the line L, is mapped to the line L', the locus of s'.

If p' is fixed as an exterior point of *E*, then the same argument gives the locus of s' to be the image of the chord [tu], that is, the chord of the ellipse, [t'u']. **QED**

4 : Similarity transformations

In our earlier discussion on the behaviour of distance under affine transformations, we examined this case:

Let **f** be an affine transformation of the plane $\prod$ which has the co-ordinate form

$$(x,y) \to (x',y') \quad \text{where}$$

$$x' = ax + by + k_1$$
$$y' = cx + dy + k_2$$

$$\text{and} \quad ad - bc \neq 0 \quad ,$$

and let p_1 and p_2 be distinct points with coordinates (x_1,y_1) and (x_2,y_2) respectively. Let $|p_1p_2| = r$ and let the half-line $[p_1p_2$ have angle of inclination θ .

We proceeded to establish an expression for the length of $[p_1'p_2']$, the image of $[p_1p_2]$, and found that the ratio of the length of the image segment to that of the original segment was the **magnification ratio k**, where

$$k = \sqrt{(a\cos\theta + b\sin\theta)^2 + (c\cos\theta + d\sin\theta)^2}$$

We concluded from this that, in general, the magnification ratio k would vary with both the affine transformation **f** (because of a,b,c and d) and the angle θ.

In this section, we now turn to the consideration of those affine transformations where the magnification ratio k varies only with **f** and where the angle of inclination of lines plays no role in fixing the value of k. Such transformations are called **similarity transformations**.

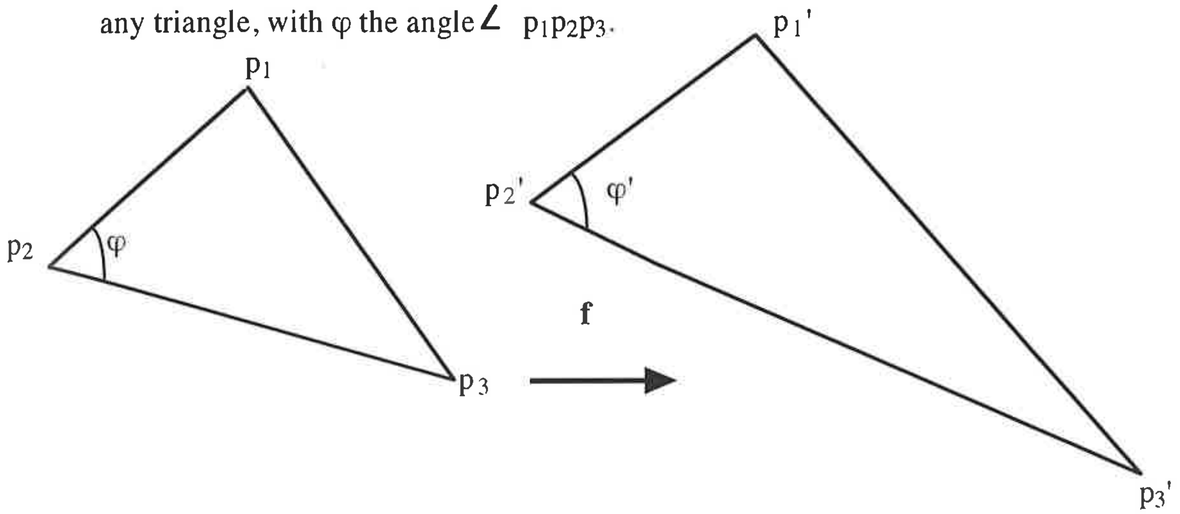
We will establish six results:

each similarity transformation **f** maps each angle φ onto an equal angle	**4.1**
the image under a similarity transformation **f** of any triangle is a similar triangle	**4.2**
the circumcentre of a triangle is mapped to the circumcentre of the image triangle under a similarity transformation **f**	**4.3**
the orthocentre of a triangle is mapped to the orthocentre of the image triangle under a similarity transformation **f**	**4.4**
the incentre of a triangle is mapped to the incentre of the image triangle under a similarity transformation **f**	**4.5**
the image under a similarity transformation **f** of a circle is a circle	**4.6**

Result 4.1: Each similarity transformation **f** maps each angle φ onto an equal angle.

Proof 4.1

Let **f** be a similarity transformation, with constant magnification ratio k, and $\Delta p_1p_2p_3$ is any triangle, with φ the angle $\angle\ p_1p_2p_3$.



Applying the cosine rule to the triangle $\Delta p_1p_2p_3$ we obtain:

$$|p_2p_3|^2 = |p_1p_2|^2 + |p_1p_3|^2 - 2|p_1p_2|.|p_1p_3|.\cos\varphi$$

Applying the cosine rule to the image triangle $\Delta p_1'p_2'p_3'$ we obtain:

$$\left|p_2'p_3'\right|^2 = \left|p_1'p_2'\right|^2 + \left|p_1'p_2'\right|^2 - 2\left|p_1'p_2'\right|.\left|p_1'p_3'\right|.\cos\varphi' \quad (*)$$

As the magnification ratio k is constant for a similarity transformation, we see that

$$\left|p_2'p_3'\right| = k|p_2p_3|,\ \left|p_3'p_1'\right| = k|p_3p_1|,\ \text{and}\ \left|p_1'p_2'\right| = k|p_1p_2|.$$

Substituting for these expressions in (*) gives:

$$k^2|p_2p_3|^2 = k^2|p_1p_2|^2 + k^2|p_1p_3|^2 - 2k|p_1p_2|.k|p_1p_3|.\cos\varphi'$$

Dividing by k^2 gives

$$|p_2p_3|^2 = |p_1p_2|^2 + |p_1p_3|^2 - 2|p_1p_2|.|p_1p_3|.\cos\varphi'$$

From which we conclude that $\cos\varphi = \cos\varphi'$

and consequently $\varphi = \varphi'$ as $0 \le \varphi \le 180$.

Thus each similarity transformation **f** maps each angle φ onto an equal angle. **QED**

Recall that two triangles are said to be **similar** if their corresponding angles are equal in size.

The application of Result **4.1** gives **4.2** immediately.

Result 4.2: The image under a similarity transformation **f** of any triangle is a similar triangle.

Proof 4.2

Let **f** be a similarity transformation, with constant magnification ratio k, and $\Delta p_1p_2p_3$ is any triangle. Let the angles of $\Delta p_1p_2p_3$ be φ, ψ and χ.

As **f** is an affine transformation, the triangle $\Delta p_1p_2p_3$ is mapped to a triangle $\Delta p_1'p_2'p_3'$ and the angles φ, ψ and χ are mapped to φ', ψ' and χ' respectively.

By **4.1**, $\varphi=\varphi'$, $\psi=\psi'$ and $\chi=\chi'$.

Thus $\Delta p_1p_2p_3$ and $\Delta p_1'p_2'p_3'$ are similar triangles. **QED**

We have already established above that each affine transformation has an inverse which is also an affine transformation. We now prove that each similarity transformation has an inverse which is also a similarity transformation.

Let S denote the set of all similarity transformations of the plane $\prod$.

If **f** ε S has a magnification ratio $k \neq 0$ and $p' = \mathbf{f}(p)$ for all $p \varepsilon S$, then

$$|p_1'p_2'| = k\,|p_1p_2|.$$

Hence for all $p_1', p_2' \,\varepsilon \prod$,

$$\frac{1}{k}\left|p_1'p_2'\right| = |p_1p_2| = \left|\mathbf{f}^{-1}\left(p_1'\right)\mathbf{f}^{-1}\left(p_2'\right)\right|.$$

As **f** is an affine transformation, so is $\mathbf{f}^{-1}$.
As $\mathbf{f}^{-1}$ has a magnification ratio of 1/k, then it is also a similarity transformation.

We also note that the ratio of two distances remains invariant under a similarity transformation.

If **f** is a similarity transformation with magnification ratio k and $p_1 \neq p_2$, $p_3 \neq p_4$, then

$$\left|p_1'p_2'\right| = k|p_1p_2| \text{ and } \left|p_3'p_4'\right| = k|p_3p_4|$$

and so

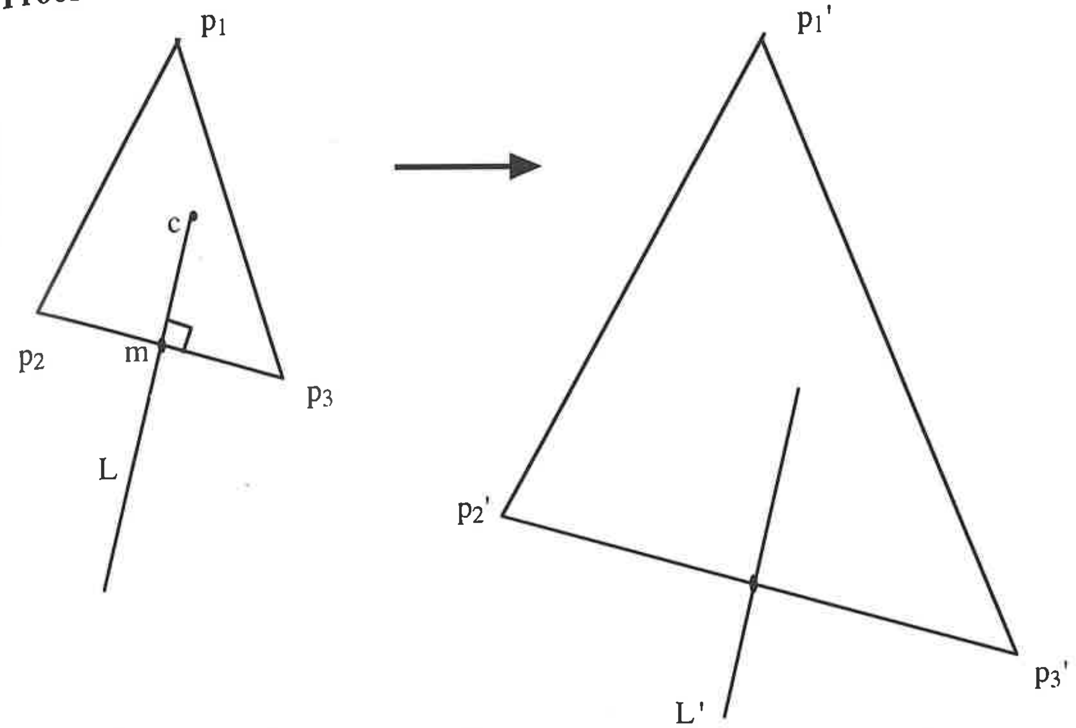
$$\frac{\left|p_1'p_2'\right|}{\left|p_3'p_4'\right|} = \frac{k|p_1p_2|}{k|p_3p_4|} = \frac{|p_1p_2|}{|p_3p_4|}$$

We are now able to prove **4.3**, **4.4** and **4.5**.

Recall that the circumcentre of a triangle is the point where the perpendicular bisectors of the sides of the triangle meet. This point is the centre of the circle which circumscribes the triangle.

Result 4.3: The circumcentre of a triangle is mapped to the circumcentre of the image triangle under a similarity transformation **f**

Proof 4.3



If p_1, p_2, and p_3 are non-collinear points, let c be the circumcentre of $\Delta\ p_1\ p_2\ p_3$.
If L is the perpendicular bisector of the side $[\ p_2\ p_3]$, then L goes through the midpoint m of $[\ p_2\ p_3]$ and is at right-angles to $p_2\ p_3$.
As **f** is an affine transformation, then

the line L is mapped to a line L'

as $m\ \varepsilon\ L$ then $m'\ \varepsilon\ L'$

as m is the mid-point of $[\ p_2\ p_3]$ then m' is the midpoint of $[\ p_2'\ p_3']$.

As **f** is a similarity transformation, then

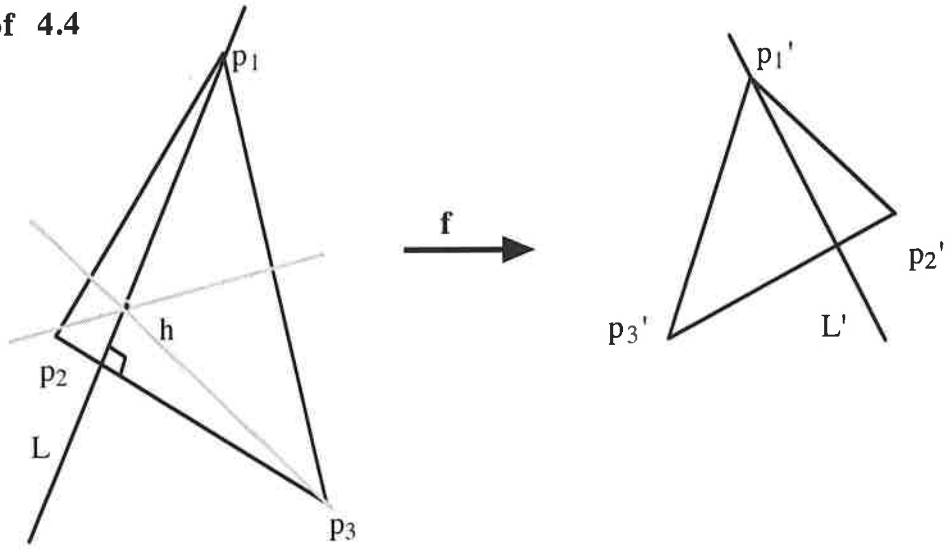
the right-angle between L and $[\ p_2\ p_3]$ is mapped to the angle between L' and $[\ p_2'\ p_3']$ and this angle will also be a right-angle. **(4.1)**

Thus the perpendicular bisector of $[\ p_2p_3]$ is mapped to the perpendicular bisector of $[\ p_2'\ p_3']$.
Similar results may be established for the perpendicular bisectors of the other two sides.
As the circumcentre c is the intersection of the perpendicular bisectors of the triangle $\Delta\ p_1\ p_2\ p_3$, then its image c' is the point of intersection of the perpendicular bisectors of the triangle $\Delta\ p_1'\ p_2'\ p_3'$, that is, c' is the circumcentre of the triangle $p_1'\ p_2'\ p_3'$. **QED**

Recall that the orthocentre of a triangle is the point where the altitudes of the triangle meet. An altitude is the line running through a vertex and its opposite side and perpendicular to that side.

Result 4.4: The orthocentre of a triangle is mapped to the orthocentre of the image triangle under a similarity transformation **f**.

Proof 4.4



Let h be the orthocentre of the triangle $\Delta\, p_1\, p_2\, p_3$. Let L be the line through the vertex p_1 which is perpendicular to the opposite side $p_2\, p_3$.
As **f** is an affine transformation, then

the line L is mapped to a line L'

as $p_1 \varepsilon$ L then $p_1' \varepsilon$ L'.

As **f** is a similarity transformation, then

if L is at right-angles to $p_2\, p_3$ then L' is at right-angles to $p_2'\, p_3'$. **(4.1)**

Thus the perpendicular from a vertex to the opposite side-line of $\Delta\, p_1\, p_2\, p_3$ maps into a perpendicular from a corresponding vertex to the opposite side-line of $\Delta\, p_1'\, p_2'\, p_3'$.

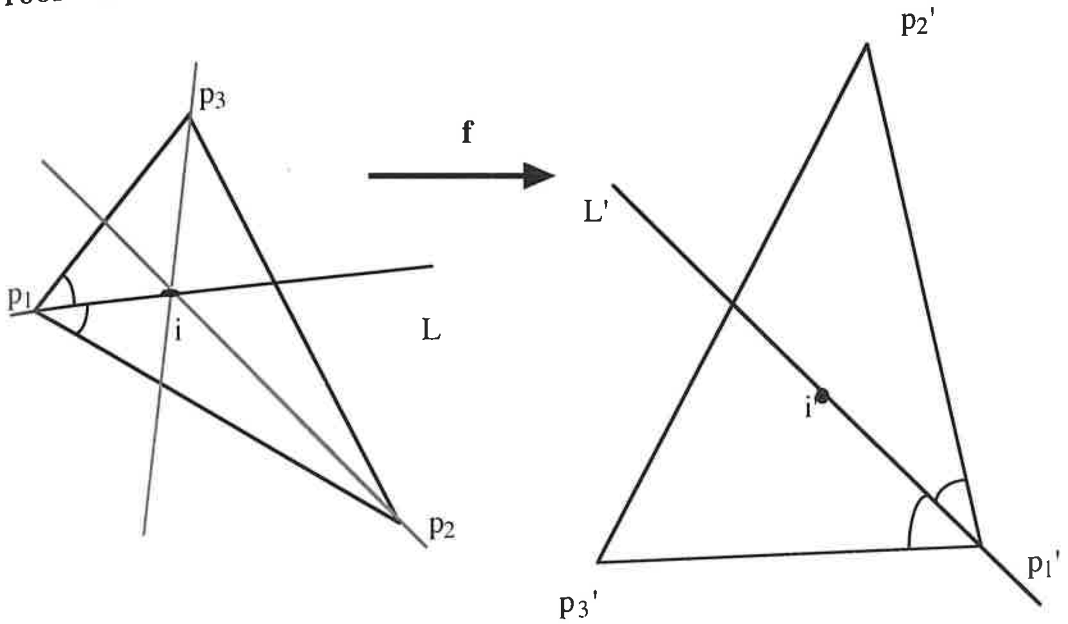
Similar results may be established for the other two such perpendiculars.

As the orthocentre h is the intersection of the three such perpendiculars of $\Delta\, p_1\, p_2\, p_3$, then its image h' is the intersection of the corresponding three such perpendiculars of $\Delta\, p_1'\, p_2'\, p_3'$, that is, h' is the orthocentre of the triangle $\Delta\, p_1'\, p_2'\, p_3'$. **QED**

Recall now that the incentre of a triangle is the point where the lines bisecting the three interior angles of a triangle meet. This point is the centre of the circle which is inscribed in the triangle.

Result 4.5: The incentre of a triangle is mapped to the incentre of the image triangle under a similarity transformation **f**.

Proof 4.5



Let i be the incentre of the triangle $\Delta p_1p_2p_3$. If now L is the line p_1i then **f**(L) is the line $p_1'i'$.

The angles $\angle\, ip_1p_2$ and $\angle\, ip_1p_3$ are equal as L is the angle bisector. By **4.1** the angles $\angle\, i'p_1'p_2'$ and $\angle\, i'p_1'p_3'$ are equal. Hence L' is a line bisecting the angle $\angle\, p_2'p_1'p_3'$.

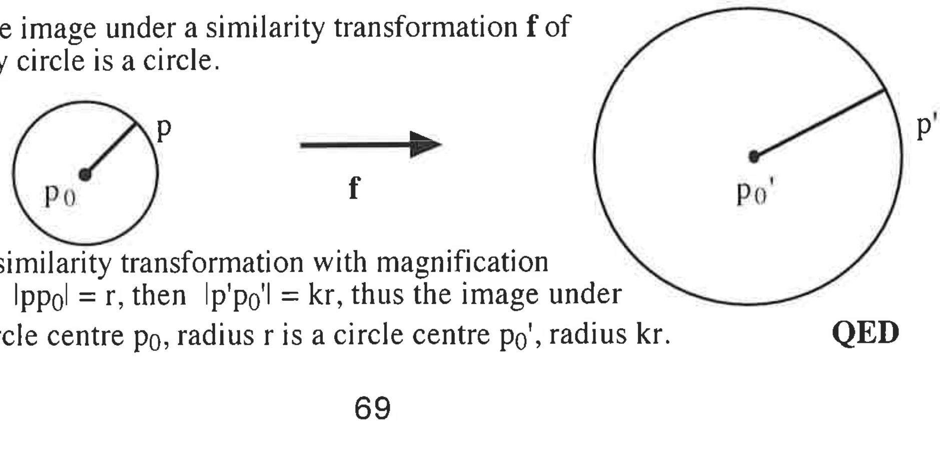
So the bisector of one corner angle of the triangle $\Delta p_1p_2p_3$ is mapped by **f** into the bisector of a corresponding corner angle of the triangle $\Delta p_1'p_2'p_3'$. Similar results may be established for the other two angle bisectors.

As the incentre i is the intersection of the three angle bisectors of the triangle $\Delta p_1p_2p_3$, then its image i' is the intersection ot the corresponding angle bisectors of the triangle $\Delta p_1'p_2'p_3'$, that is, i' is the incentre of the triangle $\Delta p_1'p_2'p_3'$. **QED**

Recall now that a circle is the locus of a point p, such that $|pp_0|$ is constant for a fixed point p_0.

Result 4.6: The image under a similarity transformation **f** of any circle is a circle.

Proof 4.6



Let **f** be a similarity transformation with magnification ratio k. If $|pp_0| = r$, then $|p'p_0'| = kr$, thus the image under **f** of the circle centre p_0, radius r is a circle centre p_0', radius kr. **QED**

As S, the set of similarity transformations, is a subset of Cl, the set of affine transformations, every affine invariant is also an invariant under similarity transformations, but not of course, vice versa. Combining affine invariants with those special invariants of similarity transformations permits the following summary of some of the invariants for similarity transformations:

the ratio of two distances
the magnitude of an angle
perpendicularity of two lines
similarity of two triangles
being an orthocentre of a triangle
being a circumcircle of a triangle
being an incentre of a triangle
being a circle
being a tangent to a circle
being a circle circumscribed to a triangle *
being a circle inscribed in a triangle *
being a square
being a rectangle
being a right-angled triangle *
being an isosceles triangle *
being an equilateral triangle *
being the bisector of an angle

(* indicates results not proven here but which may be easily obtained)

In introducing this section, it was remarked that we would consider those affine transformations where the magnification ratio k varies only with **f** and where the angle of inclination of lines plays no role in fixing the value of k. We now examine the matrices of such transformations, and seek to establish which pattern of entries in the matrix will correspond to a similarity transformation.

Recall our discussion above, where it was first established that the magnification ratio of an **affine** transformation was the number k, where

$$k = \sqrt{(a\cos\theta + b\sin\theta)^2 + (c\cos\theta + d\sin\theta)^2}$$

Squaring each side gives

$$\begin{aligned}
k^2 &= (a\cos\theta + b\sin\theta)^2 + (c\cos\theta + d\sin\theta)^2 \\
&= \left(a^2\cos^2\theta + 2ab\sin\theta\cos\theta + b^2\sin^2\theta\right) + \left(c^2\cos^2\theta + 2cd\sin\theta\cos\theta + d^2\sin^2\theta\right) \\
&= \left(a^2 + c^2\right)\cos^2\theta + \left(b^2 + d^2\right)\sin^2\theta + (ab + cd)(2\sin\theta\cos\theta) \\
&= \left(a^2 + c^2\right)\left(\frac{1}{2}\right)(1 + \cos 2\theta) + \left(b^2 + d^2\right)\left(\frac{1}{2}\right)(1 - \cos 2\theta) + (ab + cd)(2\sin\theta\cos\theta) \\
&= \frac{1}{2}\left(a^2 + c^2 + b^2 + d^2\right) + \frac{1}{2}\underline{\left(a^2 + c^2 - b^2 - d^2\right)}(\cos 2\theta) + \underline{(ab + cd)}\sin 2\theta
\end{aligned}$$

In dealing with similarity transformations, we wish k (and so k^2) to be independent of the value of θ. This happens if TWO conditions are fulfilled:

$$a^2 + c^2 - b^2 - d^2 = 0$$
$$ab + cd = 0$$

If these conditions are satisfied, then we see that

$$k^2 = \frac{1}{2}\left(a^2 + b^2 + c^2 + d^2\right)$$

that is, independent of θ.

We now attempt to extract some information about the matrix entries a,b,c,d from the conditions

$$a^2 + c^2 - b^2 - d^2 = 0 \quad ...*$$

$$ab + cd = 0 \quad ...**$$

Firstly we note that we cannot have $b^2 + d^2 = 0$, for then $b^2 = -d^2$. As b and d are real numbers, this means that the only solution here is b = d = 0. If this is the case then the equation marked * becomes

$$a^2 + c^2 = 0$$

which, by the same argument gives a = c = 0. But now a = b= c = d = 0, and the determinant of the matrix (ie ad-bc) must be 0, which means the transformation does not have an inverse and consequently cannot be affine.

If we multiply equation * by b^2 we obtain

$$b^2a^2 + b^2c^2 - b^2\left(b^2 + d^2\right) = 0$$

which becomes

$$b^2a^2 + b^2c^2 = b^2\left(b^2 + d^2\right) \quad ...***$$

By equation **, $ab + cd = 0$ and so $ab = -cd$.

Squaring gives $a^2b^2 = c^2d^2$.

Substituting this into equation *** gives

$$c^2d^2 + b^2c^2 = b^2\left(b^2 + d^2\right)$$

so

$$c^2\left(b^2 + d^2\right) = b^2\left(b^2 + d^2\right)$$

As we have already seen that $\left(b^2 + d^2\right) \neq 0$

we must have

$$c^2 = b^2 \quad \text{or} \quad b = \pm c$$

If we now multiply equation * by d^2 and proceed as above we conclude that

$$a^2 = d^2 \qquad \text{or} \qquad d = \pm a$$

This gives four possibilities:

$$\begin{pmatrix} a & -c \\ c & a \end{pmatrix} \text{ or } \begin{pmatrix} a & c \\ c & -a \end{pmatrix} \text{ or } \begin{pmatrix} a & c \\ c & a \end{pmatrix} \text{ or } \begin{pmatrix} a & -c \\ c & -a \end{pmatrix}$$

But equation ** requires ab + cd = 0, which for the last two matrices means that either a=0 or b=0, conditions that make these two matrices the same as the first two.

Thus we may conclude that if A is the matrix of a similarity transformation, then

$$\text{either } A=\begin{pmatrix} a & -c \\ c & a \end{pmatrix} \text{ or } A=\begin{pmatrix} a & c \\ c & -a \end{pmatrix}$$

with $\quad a^2+c^2 \neq 0.$

As we have already established that for a similarity transformation

$$k^2 = \frac{1}{2}\left(a^2+b^2+c^2+d^2\right)$$

we may conclude that

$$k^2 = \frac{1}{2}\left(a^2+c^2+c^2+a^2\right) = a^2+c^2$$

thus

$$k = \sqrt{a^2+c^2}$$

We pursue this approach further by carrying out the same sort of investigation into two other types of transformation: **isometries**, which preserve all distances, and **dilatations**, which map a line to a parallel line.

In the case of **isometries**, we may capture the essence of these transformations by noticing that the magnification ratio k, must be equal to one, and that the angle of inclination of the line of which a line segment is a part does not play a role. Thus we see that isometries are similarity transformations, with k=1.Thus the corresponding matrix must be of the type:

$$\text{either } A=\begin{pmatrix} a & -c \\ c & a \end{pmatrix} \text{ or } A=\begin{pmatrix} a & c \\ c & -a \end{pmatrix}$$

with $\quad a^2+c^2 \neq 0.$

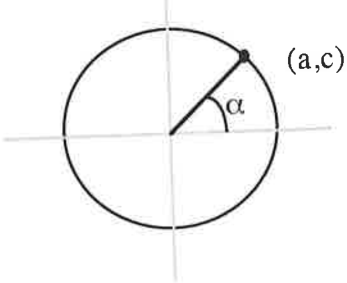
To this we add the condition that

$$k=\sqrt{a^2+c^2}=1$$

$$\Rightarrow \quad a^2+c^2=1$$

From this we see that the determinant of the matrix A must be ± 1.

It will be recalled that a useful solution to such an equation as $a^2 + c^2 = 1$ may be obtained by thinking of each solution (a,c) as a point on a unit circle.



Thus there is an angle α such that $a = \cos\alpha$ and $c = \sin\alpha$.

When the determinant of the matrix is + 1, we obtain the matrix

$$\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

and the transformation

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}$$

As the determinant is +1, this transformation preserves orientation.

In the case of $k_1 = k_2 = 0$, the transformation

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

corresponds to a **rotation** about the origin, anticlockwise through an angle α.

When the determinant is -1, orientation is reversed and we have the transformation

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}$$

In the case of $k_1 = k_2 = 0$, the transformation

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

corresponds to **axial symmetry** in the line through the origin, with angle of inclination $\alpha/2$.

As we have seen, an isometry is a similarity transformation, and consequently possesses all the invariants of that type. In addition, however, the isometries have the additional invariants of

distance
area
congruence of triangles
congruence of circles

In the case of **dilatations**, we first give a definition. An affine transformation **f** is called a **dilatation** if for each line L, we have **f**(L) ǁ L.

Let the line L have equation lx + my + n =0 and the affine transformation **f** have the matrix:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}, \quad ad - bc \neq 0.$$

We wish to obtain an expression for $\begin{pmatrix} x \\ y \end{pmatrix}$, and hence for x and y, so that on substituting for x and y in $lx + my + n = 0$, we will obtain an expression for $\mathbf{f}(\mathrm{L})$, the image of L.

We proceed as follows, using matrix algebra.

$$\begin{pmatrix} x' \\ y' \end{pmatrix} - \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \quad \text{or}$$

$$\begin{pmatrix} x' - k_1 \\ y' - k_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \qquad \text{......(*)}$$

We now calculate the inverse of the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

This is possible as $ad - bc \neq 0$.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{\det A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

We now pre - multiply each side of equation (*) by the inverse matrix.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}\begin{pmatrix} x' - k_1 \\ y' - k_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \quad \text{which becomes}$$

$$\frac{1}{\det A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} x' - k_1 \\ y' - k_2 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

From this we extract the following expressions for x and y :

$$x = \frac{1}{\det A}\left(d(x' - k_1) - b(y' - k_2)\right) = \frac{d}{\det A}x' - \frac{b}{\det A}y' + j_1$$

$$y = \frac{1}{\det A}\left(-c(x' - k_1) + a(y' - k_2)\right) = -\frac{c}{\det A}x' + \frac{a}{\det A}y' + j_2$$

where j_1 and j_2 are some numbers.

We now substitute for x and y in the equation of L: lx +my +n =0.

$$l\left(\frac{d}{\det A}x' - \frac{b}{\det A}y' + j_1\right) + m\left(-\frac{c}{\det A}x' + \frac{a}{\det A}y' + j_2\right) + n = 0$$

Multiplying through by detA, we obtain

$l(dx' - by') + m(-cx' + ay') + s = 0,$ where s is some number.

or $\quad (ld - mc)x' + (-lb + ma)y' + s = 0.$

This is the equation of **f**(L).

We will have **F**(L) || L if the slopes of the two lines are the same.

That is, $\quad \dfrac{-l}{m} = \dfrac{-(ld - mc)}{-lb + ma}$

or $\quad -l(-lb + ma) = -m(ld - mc).$

Hence $\quad bl^2 + (d - a)lm - cm^2 = 0.$

This is to hold for all lines L, and so for all values of l and m not simultaneously zero. This holds only if $b = 0$, $a = d$, $c = 0$.

So the matrix of a dilatation must have the form $A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Note that $a \neq 0$ as detA $\neq 0$.

Thus a dilatation is a transformation of the form

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} \text{ ,where } a \neq 0. \qquad ...(*)$$

It preserves or reverses orientation according as $a \succ 0$ or $a \prec 0$.

A dilatation **f** for which there is a point p_1, and scale factor a such that $\mathbf{f}(p_1) = p_1$, is called an **enlargement with centre p_1, and scale factor a**. For such a point p_1 we will have:

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}$$

So by subtraction from (*) we have

$$\begin{pmatrix} x' \\ y' \end{pmatrix} - \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} - a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

and so

$$\begin{pmatrix} x' - x_1 \\ y' - y_1 \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x - x_1 \\ y - y_1 \end{pmatrix}$$

where $a \neq 0$. It preserves or reverses orientation according as $a \succ 0$ or $a \prec 0$.

PART B

MISCELLANEOUS

TOPICS

FOUNDATION LEVEL

ORDINARY AND HIGHER LEVEL
CORE

PROBABILITY

This topic is listed in the Ordinary syllabus under the heading of Discrete Mathematics and Statistics as follows:

> Discrete probability: simple cases, with probability treated as relative frequency. For equally likely outcomes, probability = (number of outcomes of interest) / (number of possible outcomes). Examples including coin tossing, birthday distribution, card drawing (one or two cards), sex distribution, etc.

The other two courses include essentially the same material. In the Higher course core, the restriction to "one or two cards" is omitted; additional material is included, but is restricted to the option. Thus — except in the Higher course option — consideration is limited to very simple situations. The basic strategy is simple, and is based on *counting:*

- **count** the total number involved;
- **count** the number of cases in which we are interested;
- **divide** the latter by the former to obtain the probability.

Two types of examples may be distinguished. In one, data are set out in a frequency table; typically, a person or an object is to be picked at random (hence, any person or object is equally likely to be chosen); and the probability of the person or object having certain characteristics is to be obtained. In the other, the outcomes of an action such as tossing two coins have to be worked out, and the probability of particular outcomes determined.

Data given via a frequency distribution

A typical easy example to introduce the ideas might be as follows. There are 25 students in a class; 20 are right-handed and 5 are left-handed. If a member of the class is chosen at random, what is the probability that this student is left-handed?

The reasoning proceeds thus:

- there are 25 students altogether;
- 5 of them are the ones in which we are interested (the left-handed ones);
- the probability is therefore 5/25, or 0.2.

More complicated examples might involve two criteria (say, eye colour as well as "handedness"). Data may be presented as a two-way frequency table.

Consideration of outcomes

Students can proceed to consider the probability of obtaining certain outcomes from an action such as tossing fair coins or throwing fair dice. Since all outcomes are equally likely, the main strategy is again based on *counting,* but this time with a preliminary stage involving *listing:*

- **list and count** the possible outcomes;
- **list and count** the outcomes of interest;
- **divide** to get the probability.

A couple of short cuts are introduced as the work progresses, to obviate tedious listing and lengthy counting.

To clarify thinking, it may be necessary first to establish what an outcome looks like. Thus, examples can be tackled as follows:

(i) What is a **typical outcome**? For instance,
- for throwing a die, it is say 4;
- for tossing two coins, it is say (T,H);
- for a three-child family, it is say (G, G, B).

(ii) What are the **possible outcomes**? For instance:
- for throwing two dice, they are:

(H,H), (H,T), (T,H), (T,T).

How many are there?
- in this case, 4.

(iii) What are the **outcomes of interest**?

Consider, say, "a head and a tail" — this means without regard to order (contrast "a head and *then* a tail"), so they are:

(H,T), (T,H)

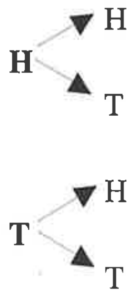
How many are there?
- in this case, 2.

(iv) **Divide**:

2/4, or 0.5.

The result can be **interpreted** for instance as "a fifty-fifty chance", "an even-money chance", or "as likely as not to happen".

In the particular example given here, students often have difficulty in accepting that (H,T) and (T,H) are different outcomes, and that the *four* outcomes listed are equally likely. Use of coins of different value, or emphasis on "first toss, second toss", may clarify the former point. For the latter, it may help to ask each member of the class to toss two coins and then to collect the frequencies. To reduce the likeliihood of obtaining an atypical distribution, three or four "rounds" of tossing may be carried out. (It may be noted that expected value is not actually on the course, but students seem to find little difficulty with an intuitive connection between this and probability.) Other methods include the use of tree diagrams:



(**H**,H)

(**H**, T)

(**T**, H)

(**T**, T)

and also "tables":

(H, H) (H, T)
(T, H) (T, T).

The latter approach is also useful for presenting the 36 outcomes that occur when two dice are thrown. These can be set out in a rectangular array, for instance as follows:

(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

Occurrences of (say) "a total of 5" — (1, 4), (2, 3), (3, 2), and (4, 1) — form a diagonal line, and so are easy to identify. Equally, questions about "a total of 4 *or less*" or "a total of *less than* 5" have solutions that are easily picked out from the array (by looking at the top left-hand corner). The same is true for questions involving say "*at least* one 2" or "*exactly* one 2".

Short cuts

As indicated above, the basic "listing and counting" strategy becomes difficult for large sets of outcomes. Hence, as a **first** short cut, a method is needed for finding the size without listing and counting. This is where the Fundamental Principle of Counting (FPC) comes into play. (In fact the principle was included into the course primarily so that it would be available for use in such cases, though it is equally valuable in the — related —area of permutations and combinations, and is of interest also in its own right.)

The FPC can be introduced via examples such examples such as: a menu with three first courses and four desserts gives how many choices of two-course meal? If you have two pairs of jeans and five shirts, how many (sensible) outfits can you make?

When applied to sets of possible outcomes, we obtain:

- a die and a coin: 6x2, hence 12 possible outcomes (checked by listing them);

- three coins: 2x2x2, hence 8 (which again can be checked by listing, albeit with a little more difficulty);
- three dice ...a case in which the outcomes cannot easily be listed! ... 6x6x6, hence 216 outcomes.

In cases with large numbers of possible outcomes, identifying outcomes of interest would usually be restricted to simple cases.

A **second** short cut, this time for outcomes of interest, can be used when these appear to be numerous but the outcomes *not* of interest are easily identified. Students can be encouraged to "creep up on the answer from behind". Thus, they can approach the problem as follows:

- **count** outcomes NOT of interest;
- **subtract** from the total number of outcomes.

Note

In the interests of keeping the courses appropriately short, the emphasis was put on material that could be handled by listing and counting or by methods immediately derived from this approach. Unless they are taking the Higher course option, students are **not** expected to know the standard results for $p(A \cup B)$, $p(A).p(B)$, or $p(A')$.

HIGHER LEVEL

CORE

THE FACTOR THEOREM

The Factor Theorem is listed in the syllabus under the heading of Algebra as :

*The Factor Theorem for polynomials of degree two or three.

The asterisk indicates that a proof of this result may be examined.

This theorem has been traditionally approached using the Remainder Theorem, a method that is unsatisfactory in that it leaves out the only appropriate case when x=a.

If f is a quadratic or cubic polynomial, so that

$f(x) = a_3x^3 + a_2x^2 + a_1x + a_0$, for all $x \in \mathbf{R}$,

then for any $k \in \mathbf{R}$, $x - k$ is a factor of $f(x) - f(k)$.

For, for all $x \in \mathbf{R}$,

$$
\begin{aligned}
f(x) - f(k) &= a_3x^3 + a_2x^2 + a_1x + a_0 - \left[a_3k^3 + a_2k^2 + a_1k + a_0\right] \\
&= a_3\left(x^3 - k^3\right) + a_2\left(x^2 - k^2\right) + a_1(x - k) + a_0 - a_0 \\
&= a_3(x - k)\left(x^2 + kx + k^2\right) + a_2(x - k)(x + k) + a_1(x - k) \\
&= (x - k)\left[a_3\left(x^2 + kx + k^2\right) + a_2(x + k) + a_1\right] \\
&= (x - k)\left[a_3x^2 + \left(a_3k + a_2\right)x + a_3k^2 + a_2k + a_1\right].
\end{aligned}
$$

As $a_3x^2 + \left(a_3k + a_2\right)x + a_3k^2 + a_2k + a_1$ is a polynomial, this shows that $(x - k)$ is a factor of $f(x) - f(k)$.

In particular, if $f(k) = 0$ then $x - k$ is a factor of $f(x)$.

THE PERPENDICULAR DISTANCE FROM A POINT TO A STRAIGHT LINE

This result is listed in the syllabus under the heading of Geometry as:

*Length of a perpendicular from (x_1, y_1) to $ax + by + c = 0$.

The asterisk indicates that a proof of this result may be examined.

The approach given here is more direct than many other approaches.

Consider the line L with equation $ax + by + c = 0$ and the point p_1 with coordinates (x_1, y_1). Then the perpendicular distance from p_1 to L is

$$\frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}.$$

Proof : -

Let N be the line such that $L \perp N$ and $p_1 \in N$. Now a line with equation $-bx + ay + e = 0$ is perpendicular to L; for it to pass through p_1 we need $-bx_1 + ay_1 + e = 0$, so that $e = bx_1 - ay_1$. Thus N has as equation

$$-bx + ay + bx_1 - ay_1 = 0.$$

To find the coordinates (x, y) of the point p of intersection of L and N we would need to solve simultaneously the equations

$$ax + by = -c,$$
$$-bx + ay = -bx_1 + ay_1.$$

However as we wish to apply the distance formula, it is $(x - x_1)^2 + (y - y_1)^2$ that we shall actually use, and it is easier to work directly with it. We rewrite the equations as

$$a(x - x_1) + b(y - y_1) = -(ax_1 + by_1 + c),$$
$$-b(x - x_1) + a(y - y_1) = 0.$$

Now on squaring each of these and adding, we find that

$$a^2(x - x_1)^2 + b^2(y - y_1)^2 + 2ab(x - x_1)(y - y_1) + b^2(x - x_1)^2$$
$$+a^2(y - y_1)^2 - 2ab(x - x_1)(y - y_1)$$
$$= (ax_1 + by_1 + c)^2 + 0^2,$$

continued overleaf

and so

$$\left(a^2+b^2\right)\left[\left(x-x_1\right)^2+\left(y-y_1\right)^2\right]=\left(ax_1+by_1+c\right)^2.$$

Thus $\left(a^2+b^2\right)|p_1p|^2=\left(ax_1+by_1+c\right)^2$, and the result follows on dividing across by a^2+b^2 and then taking square roots.

THE DERIVATIVE OF √x.

This result is listed in the syllabus under the heading of Functions and Calculus as:

*Derivations from first principles of x^2, x^3,

$\sin x$, $\cos x$, $\sqrt{x}$ and $\frac{1}{x}$.

The asterisk indicates that a proof of this result may be examined.

The approach given here establishes the step

$$\lim_{h\to 0}\sqrt{x+h}=\sqrt{x}$$

rather than simply assuming it.

Let $x>0$. Then for $h\neq 0$ such that $x+h>0$,

$$(*)\quad \sqrt{x+h}-\sqrt{x}=\left(\sqrt{x+h}-\sqrt{x}\right)\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}$$

$$=\frac{(x+h)-x}{\sqrt{x+h}+\sqrt{x}}$$

$$=\frac{h}{\sqrt{x+h}+\sqrt{x}}.$$

Hence

$$\left|\sqrt{x+h}-\sqrt{x}\right|=\frac{|h|}{\sqrt{x+h}+\sqrt{x}}$$

$$<\frac{|h|}{\sqrt{x}},\ \text{as}\ \sqrt{x+h}>0.$$

Thus

$$\lim_{h\to 0}\left|\sqrt{x+h}-\sqrt{x}\right|=0$$

and so

$$\lim_{h\to 0}\sqrt{x+h}=\sqrt{x}.$$

But by (*)

$$\frac{\sqrt{x+h}-\sqrt{x}}{h}=\frac{1}{\sqrt{x+h}+\sqrt{x}}$$

so

$$\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}=\frac{1}{2\sqrt{x}}.$$

Hence

$$\frac{d}{dx}\sqrt{x}=\frac{1}{2\sqrt{x}},\ \text{for all}\ x>0.$$

HIGHER LEVEL
CORE

SOLUTION SETS OF INEQUALITIES

This topic is listed in the syllabus under the heading of Algebra as:

> Inequalities: solution of inequalities of the form
> $g(x) < k$, $x \in \mathbf{R}$, where $g(x) = ax^2 + bx + c$ or
> $g(x) = \dfrac{ax+b}{cx+d}$.
> Use of the notation $|x|$; solution of $|x-a| < b$.

The approach given here offers a systematic method of obtaining full solutions.

Example. Find explicitly the set

$$A = \left\{ x \in \mathbf{R} : \frac{7x-13}{5x-15} > 1 \right\}.$$

Solution. On bringing all the terms to one side, we see that

$$A = \{ x \in \mathbf{R} : f(x) > 0 \}$$

where

$$\begin{aligned} f(x) &= \frac{7x-13}{5x-15} - 1 \\ &= \frac{2x+2}{5x-15} \\ &= \frac{2(x+1)}{5(x-3)}. \end{aligned}$$

Let E be the set of points x such that either $f(x) = 0$ or $f(x)$ is undefined through having 0 in a denominator.

Then

$$E = \{-1, 3\}$$

and we write $\mathbf{R} \setminus E$ as a union of intervals

$$(-\infty, -1) \cup (-1, 3) \cup (3, \infty).$$

We form the following table, using for the final row the fact that the product or quotient of an even number of negative terms and any numberof positive terms is positive,while the product or quotient of an odd number of negative terms and any number of positive terms is negative.

	x < -1	-1 < x < 3	x > 3
2(x+1)	neg	pos	pos
5(x-3)	neg	neg	pos
f(x)	pos	neg	pos

From this we see that

$$A = (-\infty, -1) \cup (3, \infty).$$

HIGHER LEVEL
CORE

FINITE DIFFERENCE EQUATIONS

This topic is listed in the syllabus under the heading of Discrete mathematics and statistics as:

> Difference equations:
>
> *If a and b are the roots of the quadratic equation $px^2 + qx + r = 0$, and $s_n = la^n + mb^n$ for all n, then $ps_{n+2} + qs_{n+1} + rs_n = 0$ for all n.
>
> Examples of difference equations $ps_{n+2} + qs_{n+1} + rs_n = 0$, $(n \geq 0)$, (with s_0, s_1 given) to be solved, p, q, and r being specific numbers and the quadratic equation $px^2 + qx + r = 0$ having distinct roots.

The asterisk indicates that a proof of this result may be examined.

Equations of the form

$$ps_{n+2} + qs_{n+1} + rs_n = 0, \qquad (n \geq 0),$$

with

$$s_0 = v,\ s_1 = w,$$

are called homogeneous *difference equations* with constant coefficients, and with initial conditions. Here p, q, r, v, w are given constants. We wish to solve such equations for s_n by giving an explicit formula for it.

Such equations have applications in the biological and social sciences. The first recorded example of this type was

$$s_{n+2} - s_{n+1} - s_n = 0, \quad (n \geq 0),$$

with

$$s_0 = 0,\ s_1 = 1,$$

due to Leonardo of Pisa (nicknamed Fibonacci) in 1202, in connection with the breeding of rabbits.

The approach given here first sets out the theory and then presents a typical problem with its solution.

1. Theory.

If a and b are the roots of the quadratic equation

(i) $$px^2 + qx + r = 0,$$

and

(ii) $$s_n = la^n + mb^n, \text{ for all } n,$$

then

(iii) $$ps_{n+2} + qs_{n+1} + rs_n = 0, \quad \text{for all } n.$$

For then

$$s_{n+2} = la^{n+2} + mb^{n+2} = la^2a^n + mb^2b^n,$$
$$s_{n+1} = la^{n+1} + mb^{n+1} = laa^n + mbb^n,$$

and so for all n

$ps_{n+2} + qs_{n+1} + rs_n$

$= p\left(la^2a^n + mb^2b^n\right) + q\left(laa^n + mbb^n\right) + r\left(la^n + mb^n\right)$

$= la^n\left(pa^2 + qa + r\right) + mb^n\left(pb^2 + qb + r\right)$

$= la^n(0) + mb^n(0)$

$= 0 + 0$

$= 0.$

2. Typical example.

Solve the difference equation

(iv) $$2s_{n+2} - 3s_{n+1} + s_n = 0 \qquad (n \geq 0)$$

subject to the initial conditions

(v) $$s_0 = -1, \; s_1 = 1.$$

Solution : -

The quadratic equation (i) is now

$$2x^2 - 3x + 1 = 0,$$

with roots

$$a = 1, \; b = \frac{1}{2}.$$

With this choice, (ii) becomes

(vi) $$s_n = l(1)^n + m\left(\frac{1}{2}\right)^n = l + m\left(\frac{1}{2}\right)^n,$$

and this satisfies the difference equation (iv). It remains to satisfy the initial conditions

$$l + m\left(\frac{1}{2}\right)^0 = -1,$$
$$l + m\left(\frac{1}{2}\right)^1 = 1.$$

continued overleaf

Thus we have the simultaneous equations

$$l + m = -1,$$

$$l + \frac{1}{2}m = 1,$$

with solution

$$l = 3,\ \mathrm{m} = -4.$$

Thus our solution is

$$s_n = 3 - 4\left(\frac{1}{2}\right)^n, \quad \text{for all } n \geq 0.$$

HIGHER LEVEL
CORE AND OPTION

AN APPROACH TO THE DERIVATIVES OF EXPONENTIAL AND LOGARITHMIC FUNCTIONS

These topics are listed in the syllabus under the heading of Functions and Calculus in the core section and Further Calculus and Series in the optional section.

The discussion given here is intended to be of use in introducing the derivatives of exponential and logarithmic functions. In some approaches, students are simply given the results as established facts. Other approaches use the Maclaurin expansion of e^x as a given, and the series is differentiated term by term. There are drawbacks to all of these: the appearance of results without a motivating discussion, the use of results from an option in establishing results in the core, problems with the issue of convergence in an infinite series and so on.

The discussion which follows seeks to base the differentiation of the exponential function e^x on an argument from first principles. This approach is not specified in the syllabus and is not therefore examinable material. The intention is rather to show the student that the differentiation of e^x may be approached in the same manner as that of $\sqrt{x}$ or sin x.

DISCUSSION

Powers a^b (where $a,b \in \mathbf{R}$ and $a > 0$) are introduced on the basis that they obey the laws of indices which hold for positive integer powers b.

If we keep b fixed and replace a by a variable x, we are dealing with the familiar power function

$$f_b(x) = x^b$$

which has the derivative

$$f_b^{\prime}(x) = bx^{b-1}.$$

If instead we keep a fixed and replace b by a variable x, we are dealing with an exponential function

$$g_a(x) = a^x.$$

If we seek to find the first derivative of this, we see that

$$\frac{g_a(x+h) - g_a(x)}{h} = \frac{a^{x+h} - a^x}{h}$$

$$= \frac{a^h - 1}{h} . a^x.$$

Now $$\lim_{h \to 0} \frac{a^h - 1}{h}$$

clearly must depend on a, so we write it as

$$\lim_{h \to 0} \frac{a^h - 1}{h} = k(a).$$

Then we have

$$g_a^{\prime}(x) = k(a)a^x \qquad \text{(i)}$$

and it is a question of somehow identifying $k(a)$.

Let us look at $k(a^c)$.

We have $g_{a^c}(x) = (a^c)^x = a^{cx}$

and so

$$\frac{g_{a^c}(x+h) - g_{a^c}(x)}{h} = \frac{a^{c(x+h)} - a^{cx}}{h}$$

$$= \frac{a^{cx+ch} - a^{cx}}{h}$$

$$= \frac{a^{cx+t} - a^{cx}}{t} \, . c, \text{ where } t = ch,$$

$$= \frac{a^t - 1}{t} ca^{cx}.$$

Now $\lim\limits_{h \to 0} \dfrac{a^t - 1}{t} ca^{cx} = k(a)ca^{cx}$.

Hence

$$g_{a^c}^{\prime}(x) = k(a)c \, . \, a^{cx} = ck(a)(a^c)^x$$

and so as from (i)

$$g_{a^c}^{\prime}(x) = k(a^c)(a^c)^x \qquad \text{(ii)}$$

we have

$$k(a^c) = ck(a) \qquad \text{(iii)}$$

We should like to choose a so that in (i) $k(a)$ has the simplest non - zero value, i.e. $k(a) = 1$. We approach this via (ii) and (iii).

Given an $a > 0$, let us choose c so that

$$ck(a) = 1.$$

Then with

$$e = a^c = [a]^{\frac{1}{k(a)}}, \qquad \text{(iv)}$$

we have

$$g_e^{\prime}(x) = g_e(x)$$

i.e.

$$\frac{d}{dx} e^x = e^x. \qquad \text{(v)}$$

If we do this, then by (iv) we have

$$e^{k(a)} = a,$$

and so $$k(a) = \log_e a.$$

Now (i) becomes

$$\frac{d}{dx} a^x = (\log_e a) a^x. \qquad \text{(vi)}$$

From its derivation e seems to depend on a, but in fact this is only apparent. For suppose that as well as (v) we also had

$$\frac{d}{dx} e_1^x = e_1^x.$$

Then

$$\frac{d}{dx}\left(\frac{e_1^x}{e^x}\right) = \frac{e^x \frac{d}{dx} e_1^x - e_1^x \frac{d}{dx} e^x}{e^{2x}}$$

$$= \frac{e^x e_1^x - e_1^x e^x}{e^{2x}}$$

$$= 0.$$

Thus for some constant C we have

$$e_1^x = Ce^x$$

for all $x \in \mathbf{R}$. On putting $x = 0$ in this we have $C = 1$ and then on putting $x = 1$ we have $e_1 = e$.

Now $$e^{\log_e x} = x$$

so on applying the chain rule we have

$$e^{\log_e x} \,.\, \frac{d}{dx} \log_e x = 1,$$

so $$x \,.\, \frac{d}{dx} \log_e x = 1.$$

Thus, corresponding to (v), we have

$$\frac{d}{dx} \log_e x = \frac{1}{x} \qquad (x > 0) \qquad \text{(vii)}$$

and as

$$\log_a x = \frac{\log_e x}{\log_e a}$$

corresponding to (vi) we have

$$\frac{d}{dx} \log_a x = \frac{1}{\log_e a} \,.\, \frac{1}{x}.$$

On using the chain rule, we have from (vii) that

$$\frac{d}{dx}\log_e(1+x) = \frac{1}{1+x} \quad . \qquad \text{(viii)}$$

On using this for $x = 0$, we have that

$$\lim_{h\to 0}\left(\frac{\log_e(1+h)-\log_e 1}{h}\right) = 1$$

and so as $\log_e 1 = 0$,

$$\lim_{h\to 0}\left(\frac{1}{h}\log_e(1+h)\right) = 1.$$

In particular we can take $h = \frac{1}{n}$ $(n \to \infty)$ and have

$$\lim_{n\to\infty}\left(n\log_e\left(1+\frac{1}{n}\right)\right) = 1.$$

Thus $$\lim_{n\to\infty}\left(\log_e\left(1+\frac{1}{n}\right)^n\right) = 1$$

and so $$\lim_{n\to\infty}\left(\left(1+\frac{1}{n}\right)^n\right) = e. \qquad \text{(ix)}$$

This can be used to approximate to the value of e and show that

$$e = 2.7182818284.........$$

If we reach the more advanced theory of calculus, as in the optional section of the syllabus, that for some functions,

$$f(x) = f(0) + \sum_{j=1}^{\infty}\frac{f^{(j)}(0)}{j\,!}x^j,$$

where $f^{(j)}(0)$ is the j - th derivative of f evaluated at 0, then from (v) we find that

$$e^x = 1 + \frac{x}{1\,!} + \frac{x^2}{2\,!} + \frac{x^3}{3\,!} + ... \qquad (x \in \mathbf{R})$$

and from (viii) that

$$\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ... \qquad (|x| < 1).$$

These series are the basis of the preparation of log. and antilog. tables.

If we put $x = 1$ in the first of these, we find that

$$e = 1 + \frac{1}{1\,!} + \frac{1}{2\,!} + \frac{1}{3\,!} + ...,$$

a more useful form than (ix).

HIGHER LEVEL
CORE AND OPTION

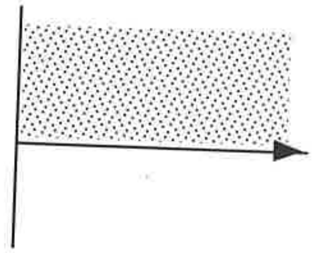
PRE-CALCULUS METHODS OF SKETCHING GRAPHS

The approach given below not only encourages the student to investigate some important properties of a function in order to produce a representative sketch of its graph, but also provides a method to achieve this.

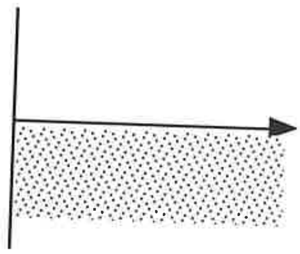
The features which are investigated for each function **f** are

1. **the behaviour of the function on $[0,\infty)$.**

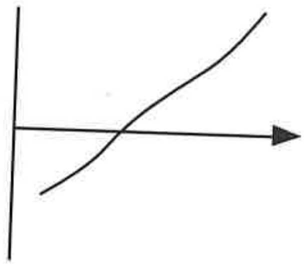
(i) In particular, if $f(x) > 0$, for all x in this interval, then **f** is a **positive function on $[0,\infty)$**;



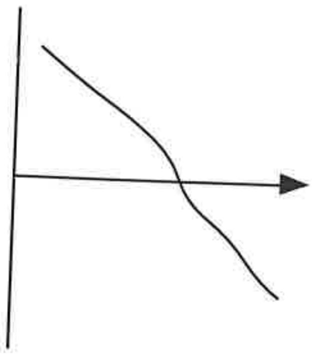
if $f(x) < 0$, for all x in this interval, then **f** is a **negative function on $[0,\infty)$**



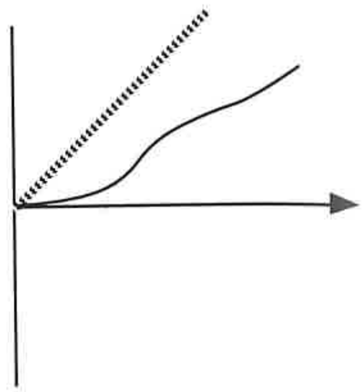
(ii) In particular, if $0 \leq p < q$ and $f(q) - f(p) > 0$, then **f** is an **increasing function on $[0,\infty)$**; (check for boundedness)



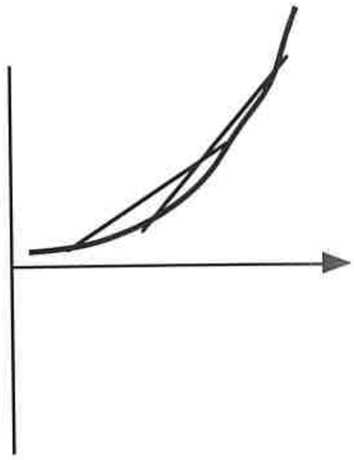
if $0 \leq p < q$ and $f(q) - f(p) < 0$, then **f** is an **decreasing function on $[0,\infty)$** (check for boundedness)



(iii) In particular, if $g(x) = ax$, $a > 0$, and $f(x) < g(x)$ for all x in this interval; then the graph of $f(x)$ will lie below that of the graph of $g(x)$, a graph well-known to students

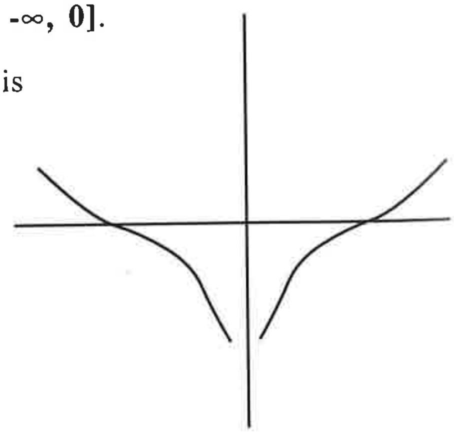


(iv) In particular, if $0 \le p < q$, and u and v are points with coordinates $(p, f(p))$, $(q, f(q))$, such that the equation of the chord uv is $y = g(x)$, and $f(x) < g(x)$, for all x in the interval (p, q), then the graph of $f(x)$ will lie below that of the graph of $g(x)$.

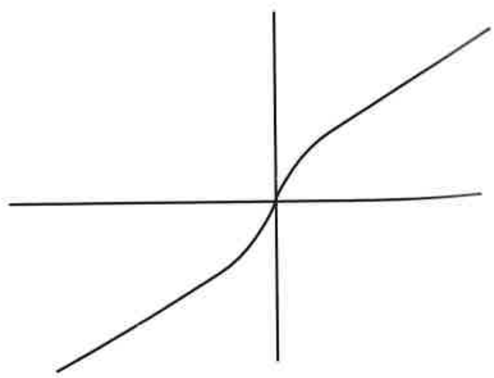


2. the behaviour of the function on $(-\infty, 0]$.

(i) Test for symmetry in the y-axis. That is, is $f(- x) = f(x)$ for all x in **R** ?



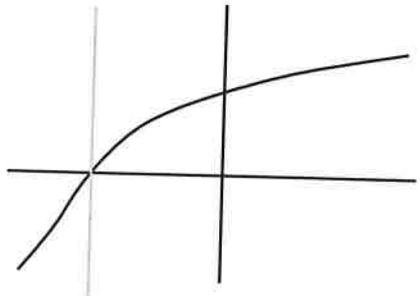
(ii) Test for symmetry in the origin (0,0). That is, is $f(-x) = -f(x)$ for all x in **R** ?



3. **the behaviour of af(x) ,**

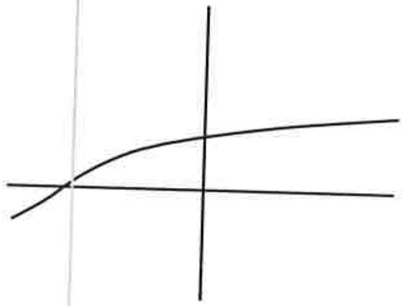
(i) **a =1**

af(x) = f(x)



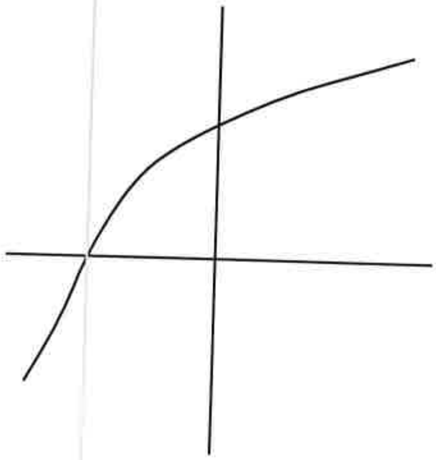
(ii) **0 < a < 1**

af(x) ≤ f(x)

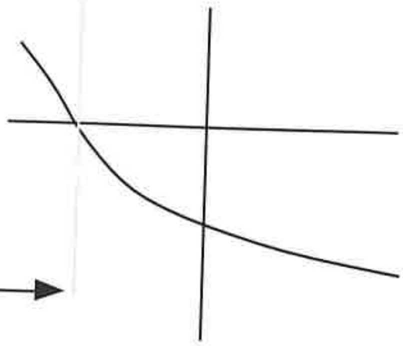


(iii) **a >1**

af(x) ≥ f(x)



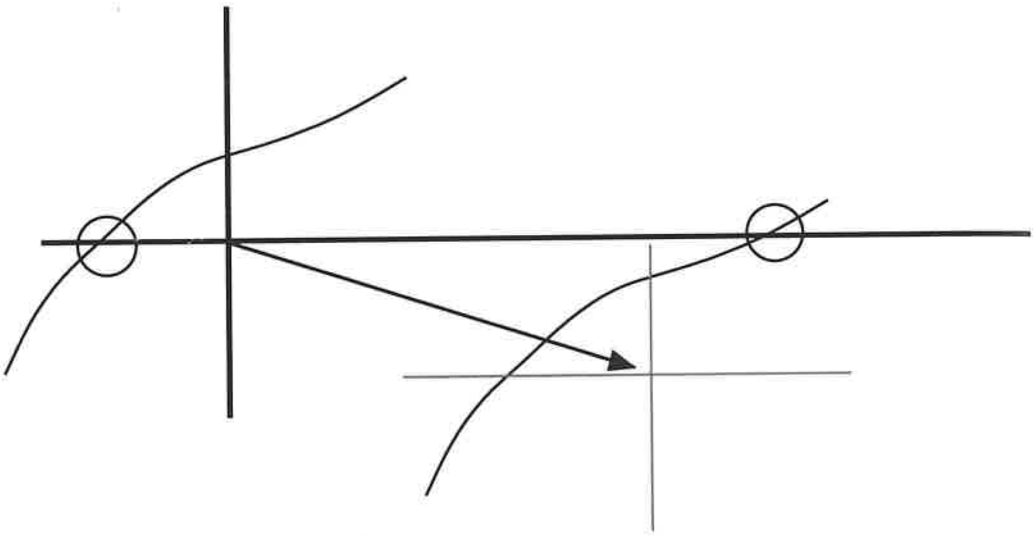
(iv) **a < 0**
Let a = -A, A > 0.
Then af(x) = - [Af(x)] , the image of Af(x) under axial symmetry in the x-axis.



note the behaviour of the root ⟶

4. the behaviour of f(x) under a translation (0,0) -> (r,s)

(i) We note that the graph $(y - s) = f(x - r)$ may be obtained from the graph $y = f(x)$ by translating the origin to the point (r,s), the x-axis (y=0) to the line y = s, and the y-axis (x=0) to the line x = r.



NB note the behaviour of the root.

Examples

Example 1.

For a fixed number $a > 0$, consider the function f defined by

$$f(x) = ax^2 \quad (x \in \mathbf{R}).$$

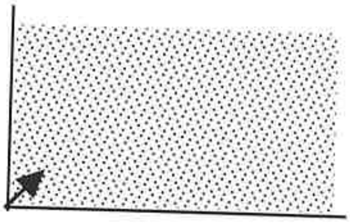
We apply the procedure set out above.

1. **the behaviour of the function on $(0,\infty)$.**

(i) In particular, if $f(x) > 0$, for all x in this interval, then **f** is a **positive function on $(0,\infty)$;** (check for bounds)

We note that

$f(0) = 0$, and $f(x) > 0$ for all $x > 0$.



(ii) In particular, if $0 \le p < q$ and $f(q) - f(p) > 0$, then **f** is an **increasing function on $[0,\infty)$**

We note that

$$f(q) - f(p) = aq^2 - ap^2 = a(q+p)(q-p).$$

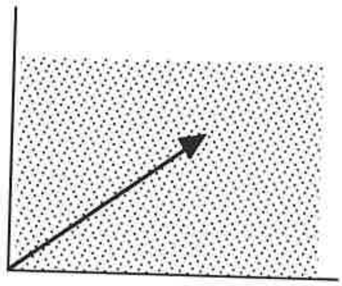
Now $a > 0$,

and $q + p > 0$, as $q, p > 0$,

and $q - p > 0$, as $q, p > 0$ and $q > p$.

Hence $a(q+p)(q-p) > 0$.

Thus f is an increasing function on $[0,\infty)$.



(iii) In particular, if $\mathbf{g}(x) = ax$, $a > 0$, and $\mathbf{f}(x) < \mathbf{g}(x)$ for all x in this interval; then the graph of $\mathbf{f}(x)$ will lie below that of the graph of $\mathbf{g}(x)$, a graph well-known to students (check for boundedness)

We note

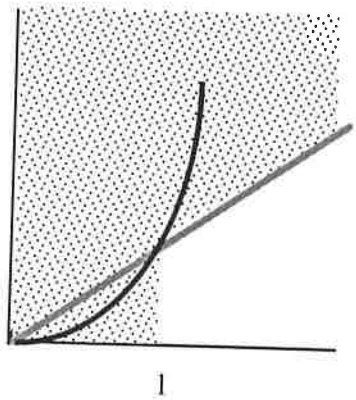
$x^2 > x$ for all $x > 1$.

So $ax^2 > ax$ for all $x > 1$.

So $f(x) > ax$ for all $x > 1$.

hence the graph of $\mathbf{f}(x)$ lies **above** that of $\mathbf{g}(x)$ for $x > 1$.

As $f(x) > ax$ and ax increases without bound as x increases, then $f(x)$ increases without bound.



(iv) In particular, if $0 \le p < q$, and u and v are points with coordinates (p, **f**(p)) , (q, **f**(q)), such that the equation of the chord uv is y = **g**(x), and **f**(x) < **g**(x), for all x in the interval (p, q), then the graph of **f**(x) will lie below that of the chord uv.

We note

If u and v are the points with coordinates (p, ap^2) and (q, aq^2), then the line uv has slope

$$\frac{aq^2 - ap^2}{q-p} = \frac{a(q-p)(q+p)}{q-p} = a(q-p), \text{ as } q \ne p.$$

Thus the line uv has the equation

$$y - ap^2 = a(q-p)(x-p)$$

or $$y = ap^2 + a(q-p)(x-p) = g(x)$$

Now $$f(x) - g(x) = ax^2 - \left[ap^2 + a(q-p)(x-p)\right]$$
$$= ax^2 - ap^2 - a(q-p)(x-p)$$
$$= a(x+p)(x-p) - a(q-p)(x-p)$$
$$= a(x-p)[x+p-(q-p)]$$
$$= a(x-p)(x-q)$$

Now $a > 0$,

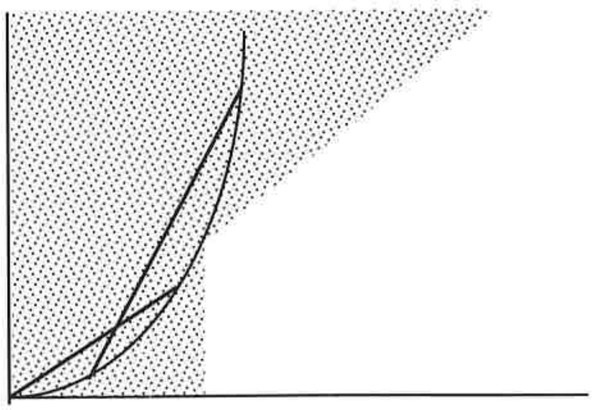
and as x lies between p and q, $p < x < q$,

so $x - p > 0$

and $x - q < 0$.

Thus $f(x) - g(x) = (+\text{ term})(+\text{ term})(-\text{ term}) < 0$.

Thus the graph of $f(x)$ lies below the chord $[uv]$ for $p < x < q$.



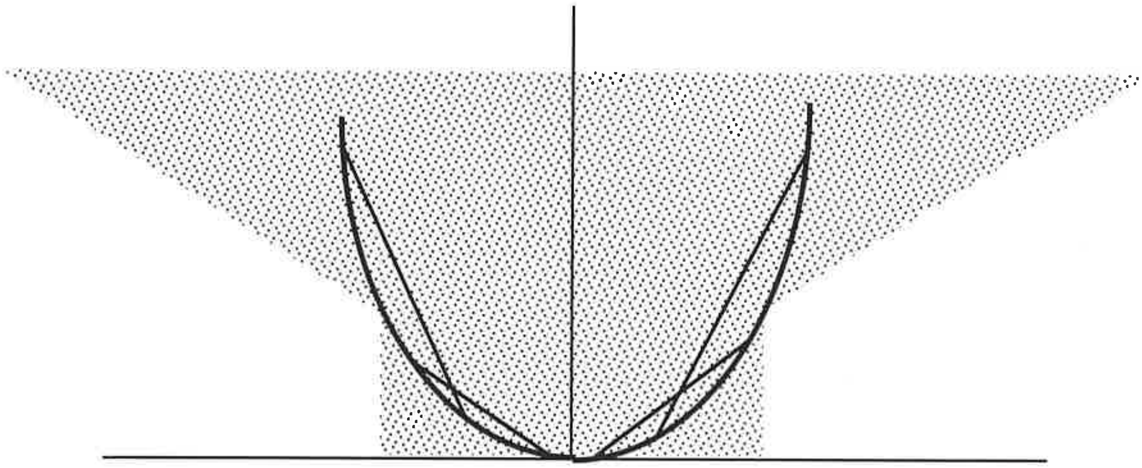
2. **the behaviour of the function on (-∞, 0].**

(i) Test for symmetry in the y-axis. That is, is **f**(- x) = **f**(x) for all x in **R** ?

We note

$$f(-x) = a(-x)^2 = ax^2 = f(x), \text{ for all } x \in \mathbf{R}.$$

So we see that the graph is symmetrical about the y - axis.

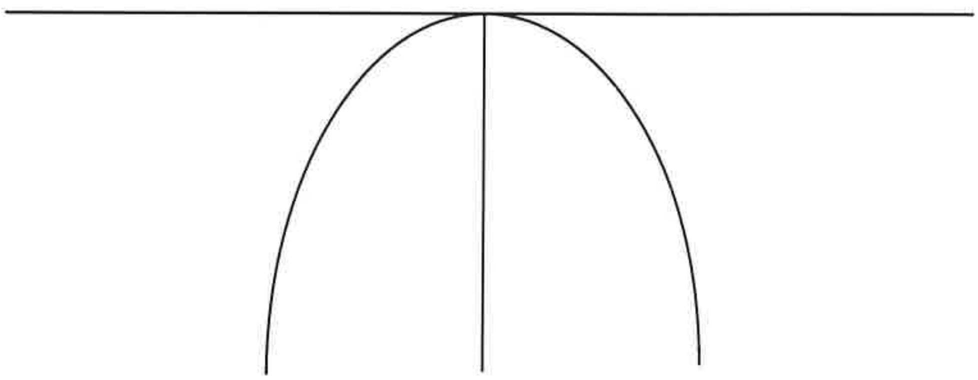


3. **the behaviour of af**(x)

We have considered the case where a > 0.
We note

If $a < 0$, then let $a = -A$, $A > 0$, and the graph $y = Ax^2$ looks like the graph shown immediately above.

As $ax^2 = -[Ax^2]$, the graph $y = ax^2$ is the image of the graph $y = Ax^2$ under axial symmetry in the x - axis.



4. the behaviour of f(x) under a translation (0,0) -> (r,s)

We note

$$ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}$$

If $y = ax^2 + bx + c$,

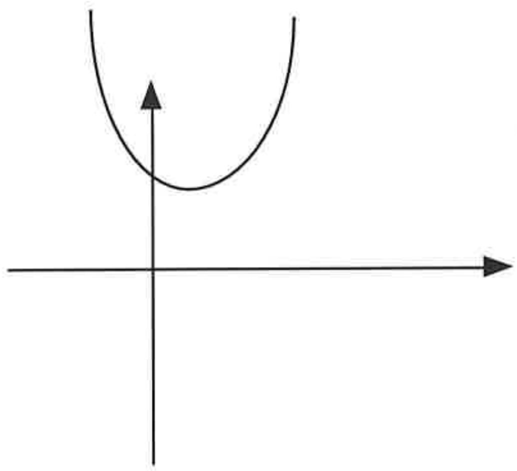
then $$y = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}$$

so $$\left(y - \frac{4ac - b^2}{4a^2}\right) = a\left(x + \frac{b}{2a}\right).$$

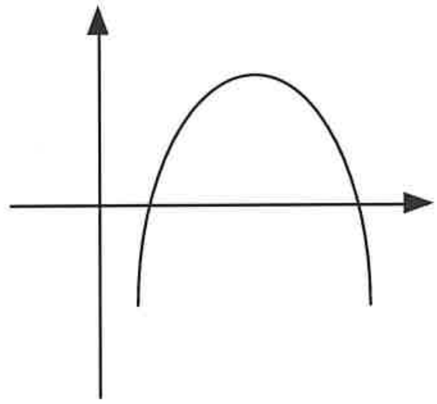
Thus the graph of $y = ax^2 + bx + c$ is the image of the graph of $y = ax^2$ under the translation

$$(0,0) \to \left(\frac{b}{2a}, \frac{4ac - b^2}{4a^2}\right).$$

We conclude that the graph of $y = ax^2 + bx + c$ has one of the shapes shown, according as $a > 0$ or $a < 0$.



a > 0



a < 0

Example 2.

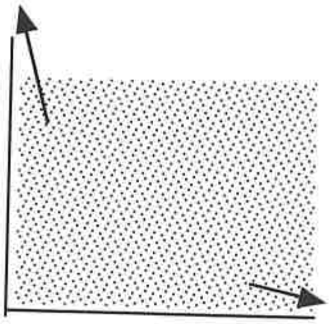
Consider the function f defined by

$$f(x) = \frac{1}{x} \text{ for all } x \neq 0.$$

We apply the procedure set out above.

1. **the behaviour of the function on $(0,\infty)$. (note $x \neq 0$)**

(i) In particular, if $f(x) > 0$, for all x in this interval, then **f** is **a positive function on $(0,\infty)$**; (check for bounds)



We note that

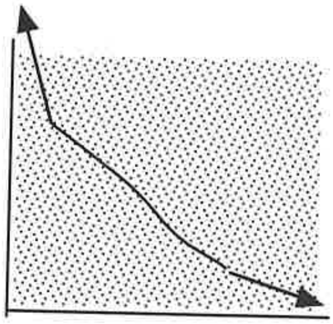
$$f(x) = \frac{1}{x} > 0 \text{ for all } x > 0.$$

As x decreases to 0, **f**(x) increases without bound.

As x increases without bound, **f**(x) becomes arbitrarily close to 0.

(ii) In particular, if $0 < p < q$ and $f(q) - f(p) > 0$, then **f** is an **increasing function on $(0,\infty)$;**

if $0 < p < q$ and $f(q) - f(p) < 0$, then **f** is a **decreasing function on $(0,\infty)$.**



If $0 < p < q$, then

$$f(q) - f(p) = \frac{1}{q} - \frac{1}{p} = -\frac{q-p}{pq} < 0,$$

thus **f** is a decreasing function on $(0,\infty)$.

(iv) In particular, if $0 < p < q$, and u and v are points with coordinates (p, **f**(p)) , (q, **f**(q)), such that the equation of the chord uv is y = **g**(x), and **f**(x) < **g**(x), for all x in the interval (p, q), then the graph of **f**(x) will lie below that of the chord uv.

If $0 < p < q$, and u,v are the points $\left(p, \frac{1}{p}\right)$ and $\left(q, \frac{1}{q}\right)$,

then the line uv has the slope $\dfrac{\frac{1}{q} - \frac{1}{p}}{q-p} = \dfrac{p-q}{qp(q-p)} = -\dfrac{1}{pq}$

and so it has the equation

$$y - \frac{1}{p} = -\frac{1}{pq}(x-p)$$

that is,

$$y = \frac{1}{p} - \frac{1}{pq}(x - p) = g(x)$$

We now examine $f(x) - g(x)$ for $0 < p < x < q$,

$$= \frac{1}{x} - \left[\frac{1}{p} - \frac{1}{pq}(x - p)\right]$$

$$= \frac{1}{x} - \frac{1}{p} + \frac{1}{pq}(x - p)$$

$$= \frac{pq}{pqx} - \frac{qx}{pqx} + \frac{x(x - p)}{pqx}$$

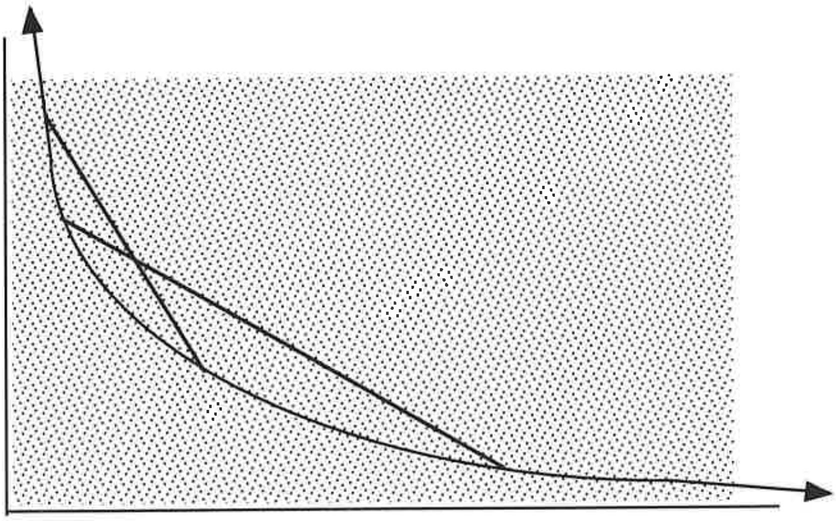
$$= \frac{q(p - x) + x(x - p)}{pqx}$$

$$= \frac{(x - p)(x - q)}{pqx}$$

As $0 < p < x < q$, $(x - p)$ is positive, $(x - q)$ is negative, pqx is positive.

Thus $f(x) - g(x) < 0$, for $0 < p < x < q$.

That is, the graph of f lies below the chord uv.



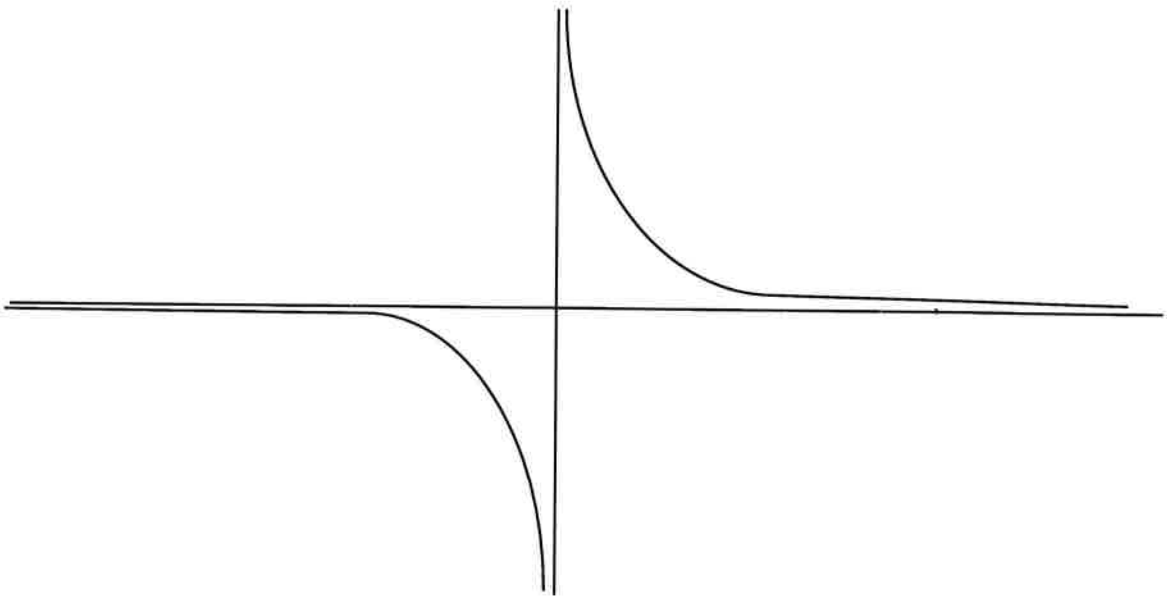
2. the behaviour of the function on $(-\infty, 0)$.

(i) Test for symmetry in the y-axis. That is, is
f(- x) = f(x) for all x in **R** ?

We note that

$$f(-x) = \frac{1}{-x} = -\frac{1}{x} = -f(x) \quad \text{for all } x \neq 0.$$

That is, the graph of **f** is not symmetrical in the y-axis, but is symmetrical about the origin.



4. the behaviour of f(x) under a translation

We see that the graph of

$$y = \frac{1}{x-a}$$

is the image of the graph of

$$y = \frac{1}{x}$$

under the translation for which (0 , 0) -> (a , 0), and so its graph has the following shape.