APPLIED MATHEMATICS – HIGHER LEVEL

Six questions to be answered.   All questions carry equal marks.

Mathematics Tables may be obtained from the Superintendent.

Take the value of g to be 9·8 metres/second^2. $\vec{i}$ and $\vec{j}$ are perpendicular unit vectors.

1. A body starts from rest at p, travels in a straight line and then comes to rest at q which is 0·696 km from p.   The time taken is 66 seconds.

For the first 10 seconds it has uniform acceleration a_1.   It then travels at constant speed and is finally brought to rest by a uniform   deceleration a_2 acting for 6 seconds.

Calculate a_1 and a_2.

If the journey from rest at p to rest at q had been travelled with no interval of constant speed, but subject to a_1 for a time t_1 followed by a_2 for time t_2, show that the time for the journey is $8\sqrt{29}$ seconds.

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2. If a body is in equilibrium under the action of two and only two forces, what can be deduced about the forces ?

If a body is in equilibrium under the action of three non parallel forces prove that their lines of action must be concurrent and that the forces may be represented in magnitude and direction by the sides of a triangle taken in order.

A straight, rigid non-uniform rod $[pq]$ of weight W rests in equilibrium inside a smooth hollow sphere of radius r. The distances of its centre of gravity, g, from p and q are 4 and 6 cm respectively.
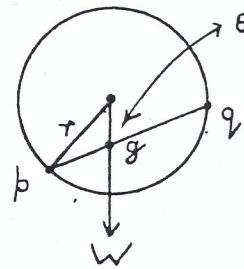
(i) Show that θ, the angle the rod makes with the vertical, is given by

$$\cos^{-1}\left(\frac{1}{\sqrt{r^2 - 24}}\right).$$

(ii) Prove that the magnitude of the reaction at p is given by

$$\frac{3\,r\,W}{5\,\sqrt{r^2 - 24}}.$$

3. Establish an expression, in terms of initial speed u and angle of inclination to the horizontal α, for the range of a projectile on a horizontal plane through the point of projection.

Deduce that the maximum range for a given u is u^2/g.

A particle is projected at initial speed u from the top of a cliff of height h, the trajectory being out to sea in a plane perpendicular to the cliff. The particle strikes the sea at a distance d from the foot of the cliff. Show that the possible times of flight can be obtained from the equation

$$g^2\,t^4 - 4\,(u^2 + gh)\,t^2 + 4\,(h^2 + d^2) = 0.$$

Hence, or otherwise, prove that the maximum value of d for a particular u and h is

$$\frac{u\,\sqrt{u^2 + 2gh}}{g}.$$

4. (i) A sphere A, mass m, moving with velocity $2u$ impinges directly on an equal sphere B, moving in the same direction with velocity u. Show that the loss in kinetic energy due to the impact is

$$\frac{m\,u^2\,(1 - e^2)}{4}$$

where e is the coefficient of restitution between the spheres.

(ii) If B had been at rest and A impinged obliquely, so that after impact, A moved with velocity $2u$ in a direction making an angle of $30°$ with the line of centres of the spheres, show that the loss in kinetic energy is three times greater than in (i).

5. Prove that the centre of gravity of a uniform triangular lamina is at the point of intersection of its medians.

A uniform triangular lamina pqr has weight W newtons.

$|pq| = 5$cm, $|qr| = 4$cm, $|pr| = 3$cm and $|\angle prq| = 90°$.

The lamina is suspended in a horizontal position by three inextensible, vertical strings, one at each vertex.

A particle, of weight $\frac{4}{3}W$ newtons is positioned on the lamina 2cm from pr and one cm from qr. Calculate the tension on each string in terms of W.
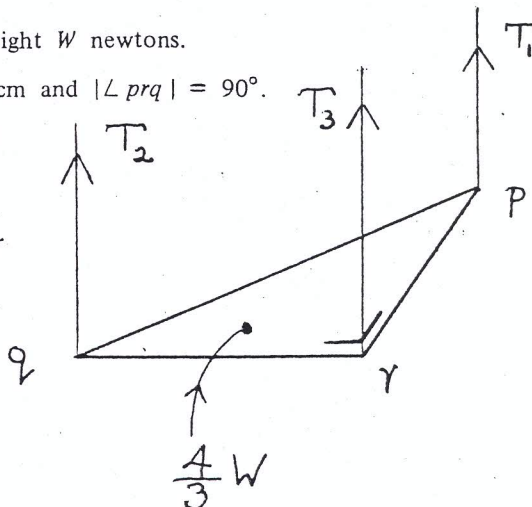
6. (a) Establish the formula $T = 2\pi\sqrt{\dfrac{l}{g}}$ for the periodic time of a simple pendulum of length l.

   The length of a seconds pendulum ($T = 2$ secs) is altered so as to execute 32 complete oscillations per minute. Calculate the percentage change in length.

   (b) A heavy particle is describing a circle on a smooth horizontal table with uniform angular velocity ω. It is partially supported by a light inextensible string attached to a fixed point 0.1 metres above the table. Calculate the value of ω if the normal reaction of the table on the particle is half the weight of the particle.

7. A uniform circular disk has mass M and radius R. Prove that its moment of inertia, I, about an axis through its centre perpendicular to its plane is $\frac{1}{2}MR^2$.

   Deduce the moment of inertia about an axis through a point on its rim perpendicular to its plane.

   A uniform circular disk has mass m and radius r. It is free to rotate about a fixed horizontal axis through a point p on its rim perpendicular to its plane. A particle of mass $2\,m$ is attached to the disk at a point q on its rim diametrically opposite p.

   The disk is held with pq horizontal and released from rest. Find, in terms of r, the angular velocity when q is vertically below p.

   If the system were to oscillate as a compound pendulum, prove that it would have a periodic time equal to that of a simple pendulum of length $\dfrac{19r}{10}$.
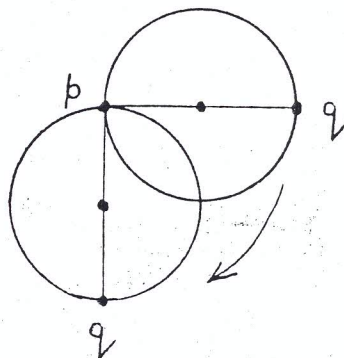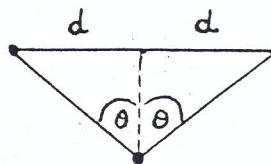
8. (a) A heavy particle is hung from two points on the same horizontal line and a distance $2d$ apart by means of two light, elastic strings of natural length l_1, l_2 and elastic constants k_1, k_2 respectively.

   In the equilibrium position the two strings make equal angles θ with the vertical.

   Prove that
   $$\sin\theta = \frac{d(k_1 - k_2)}{k_1\,l_1 - k_2\,l_2}$$

   (b) A horizontal platform, on which bodies are resting, oscillates vertically with simple harmonic motion of amplitude 0.2 m. What is the maximum integral number of complete oscillations per minute it can make, if the bodies are not to leave the platform?

9. (a) A body is weighed in water and in each of two liquids of specific gravity 0.8 and 0.75. If the resulting weights, in order, are w_1, w_2, and w_3, verify that
   $$w_1 = 5w_2 - 4w_3 .$$

   (b) A uniform rectangular board $pqrs$, $|pq| \neq |ps|$, hangs vertically in fresh water with the diagonal qs on the surface. The board is held in that position by a vertical string at p.

   (i) Show on a diagram all forces acting on the board.

   (ii) Calculate the tension (T) in the string and the buoyancy force (B) in terms of W, the weight of the board.

   (iii) Calculate the specific gravity of the board.

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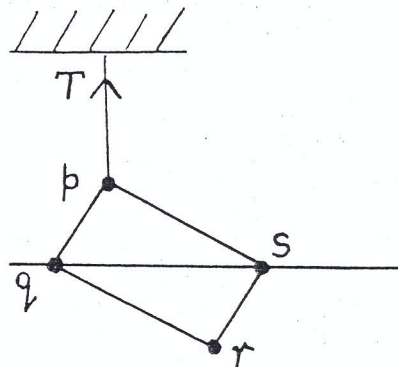
10. (a) Solve the differential equation

$$\frac{dy}{dx} + y^2 \cos^3 x = 0$$

given that $y = 2$ when $x = \pi/6$

(b) Find the general solution to

$$\frac{d^2 y}{dx^2} = K \frac{dy}{dx}$$

where K is a constant.

A particle moves in a straight line so that at any instant its acceleration is, in magnitude, half its velocity.

If its initial velocity is 3 m/s, find an expression for the distance it describes in the fifth second.