LEAVING CERTIFICATE EXAMINATION, 1980

APPLIED MATHEMATICS – HIGHER LEVEL

TUESDAY, 24 JUNE – AFTERNOON, 2.00 to 4.30

Six questions to be answered.   All questions carry equal marks.
Mathematics Tables may be obtained from the Superintendent.
Take the value of g to be 9·8 metres/second². $\vec{i}$ and $\vec{j}$ are perpendicular unit vectors.

1.   A boat has to travel by the shortest route to the point $4·25\,\vec{j}$ km and then return immediately
to its starting point at the origin.   The velocity of the water is $(8\sqrt{2}\,\vec{i} - 8\sqrt{2}\,\vec{j})$ km/hour and
the boat has a speed of 18 km/hour in still water.

If $a\vec{i} + b\vec{j}$ is the velocity of the boat on the outward journey, find a and b and the time taken
for the outward journey, leaving your answer in surd form.   Find, also, the time taken for the whole
journey.

2.   A body of weight W is supported by two vertical inextensible
strings at a and b as in diagram where $|ab| = 10$ cm.   The
tensions in the strings are T_1 and T_2 and the string of tension T_1
makes an angle of 45° with ab.   The centre of gravity of the body
is at g, the centre of $[ab]$ is c and $cg \perp ab$.

Express $|cd|$ in terms of W and T_1 and hence find the distance
of g from ab in terms of W and T_1.
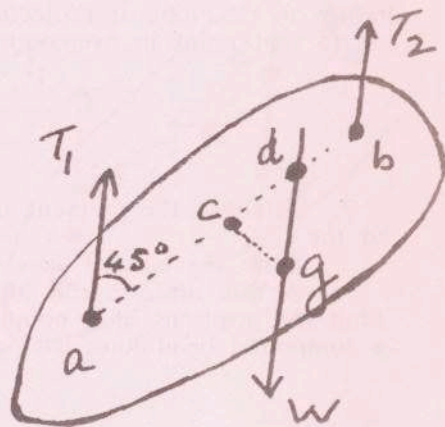
3.   A projectile is fired with initial velocity $\vec{u} = u\cos\alpha\,\vec{i} + u\sin\alpha\,\vec{j}$, where $\vec{i}$ is along the
horizontal.   A plane P passes through the point of projection and makes an angle β with the
horizontal.   If the projectile strikes the plane P at right angles to P after time t,   show that

$$t = \frac{2u\sin(\alpha - \beta)}{g\cos\beta}$$

and deduce that $2\tan(\alpha - \beta)\tan\beta = 1$.

If $\alpha - \beta = \frac{\pi}{4}$,   find in terms of u and g the range of the projectile along P.

4.   State and prove the relationship between the coefficient of friction μ and the angle of
friction λ.

The diagram shows a particle of weight W on a rough
plane making an angle α with the horizontal.   The
particle is acted upon by a force F whose line of action
makes an angle θ with the line of greatest slope.   The
particle is just on the point of moving up the plane.   Draw
a diagram showing the forces acting on the particle and
prove that

$$F = \frac{W\sin(\alpha + \lambda)}{\cos(\theta - \lambda)}.$$
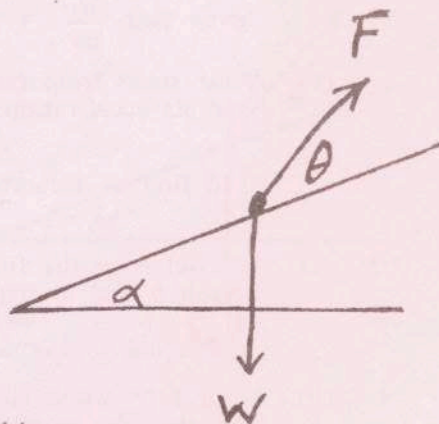
If the particle is just on the point of moving up the
plane, deduce

   (i)   the force acting up along the plane that would achieve this
   (ii)   the horizontal force that would achieve it
   (iii)   the minimum force that would achieve it.

5. (a) Two imperfectly elastic spheres of equal mass moving horizontally along the same straight line impinge and, as a result, one of them is brought to rest. Show that whatever be the value of the coefficient of restitution, $e < 1$, they must have been moving in opposite directions.

(b) A sphere A of mass m kg moving with a speed u m/s on a smooth horizontal table impinges on a smooth plane bc. This plane is inclined to the table at an angle α and the line of intersection of it with the table is at right angles to the direction of motion of the sphere. Write down the components of the velocity of A perpendicular to the plane and parallel to the plane before impact and show that $eu\sin\alpha$ is the velocity of A perpendicular to the plane after impact when e is the coefficient of restitution between the sphere and the plane. Find the magnitude of the impulse due to the impact.
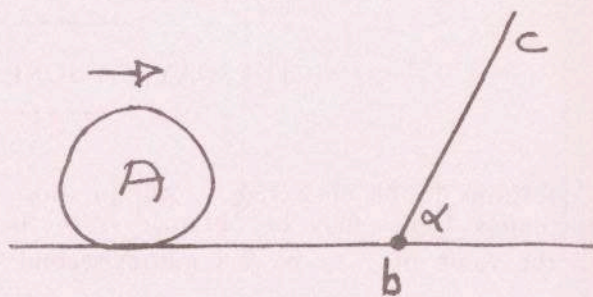
6. If a string whose elastic constant is k is stretched a distance x beyond its natural length, show that the work done is $\frac{1}{2}kx^2$.

A particle of mass m is on a rough horizontal plane and is connected to a fixed point p in the plane by a light string of elastic constant k. Initially the string is just taut and the particle is projected along the plane directly away from p with initial speed u against a constant resistance F. Find an expression for the distance x travelled by the particle. Noting that the particle will just return to its point of projection if the potential energy at any point is equal to the work done up to that point in overcoming F, show that

$$kmu^2 = 8F^2.$$

7. Establish the moment of inertia of a uniform rod about an axis through its centre perpendicular to the rod.

State the parallel axes theorem.

A thin uniform rod of length $2l$ and of mass m has a mass of $2m$ attached at its mid-point. Find the positions of a point in the rod about which the rod (with attached mass) may oscillate as a compound pendulum, having period equal to that of a simple pendulum of length l.

8. (a) A particle is moving in a straight line such that its distance x from a fixed point at time t is given by

$$x = r\cos\omega t.$$

Show that the particle is moving with simple harmonic motion.

A particle is moving in a straight line with simple harmonic motion. When it is a point p_1 of distance 0.8 m from the mean-centre, its speed is 6 m/s and when it is at a point p_2 of distance 0.2 m from the end-poisition on the same side of the mean-centre as p_1, its acceleration is of magnitude 24 m/s². If r is the amplitude of the motion, show that

$$\frac{2}{3} = \frac{r - 0.2}{r^2 - 0.64}$$

and hence find the value of r.

Find also the period of the motion and the shortest time taken between p_1 and p_2 correct to two places of decimals.

9. (a) Solve the differential equation

$$\frac{d^2y}{dx^2} + \frac{2}{y^3} = 0$$

given that $\frac{dy}{dx} = \sqrt{2}$ and $x = \sqrt{2}$ when $y = 1$.

(b) A car starts from rest. When it is at a distance s from its starting point, its speed is v and its acceleration is $5 - v^2$. Show that

$$v\,dv = (5 - v^2)\,ds$$

and find as accurately as the tables allow its speed when $s = 1.5$.

10. (a) A vessel is in the form of a frustum of a right circular cone. It contains liquid to a depth h and at that depth the area of the free surface of the liquid is $\frac{1}{4}$ of the area of the base. Find in simplest surd form the ratio of the thrust on the base due to the liquid to the weight of the liquid.

(b) A piece of wood and a piece of metal weigh 14 N and 6 N, respectively. When combined together the compound body weighs 1.9 N in water. Given that the specific gravity of the metal is 10, find the specific gravity of the wood.