1 Introduction

Let \((X, \rho)\) and \((Y, \sigma)\) be metric spaces. A function \(f : X \to Y\) is (by definition) bounded if the image of \(f\) has finite \(\sigma\)-diameter. It is well-known that if \(X\) is compact then each continuous \(f : X \to Y\) is bounded. Special circumstances may conspire to force all continuous \(f : X \to Y\) to be bounded, without \(Y\) being compact. For instance, if \(Y\) is bounded, then that is enough. It is also enough that \(X\) be connected and that each connected component of \(Y\) be bounded. But if we ask that all continuous functions \(f : X \to Y\), for arbitrary \(Y\), be bounded, then this requires that \(X\) be compact.

What about uniformly-continuous maps? Which \(X\) have the property that each uniformly-continuous map from \(X\) into any other metric space must be bounded?

We begin with an observation.

Lemma 1.1 Let \((X, \rho)\) be a metric space. Then the following are equivalent:
1. Each uniformly-continuous map from \(X\) into another metric space is bounded.
2. Each uniformly-continuous map from \(X\) into \(\mathbb{R}\) (with the usual metric) is bounded.

\(^1\)Supported by EU Research Training Network Contract no. HPRN-CT-2000-00116
Proof. Obviously (1) implies (2). The other direction follows from the facts that: (a) \( f : X \to Y \) is bounded if and only if each (or any one) of the compositions

\[ \sigma(b, \bullet) \circ f : x \mapsto \sigma(b, f(x)) \quad (b \in Y) \]

is bounded, and (b) the composition \( \sigma(b, \bullet) \) is uniformly-continuous if \( f \) is uniformly-continuous.

This allows us to concentrate on the case \( Y = \mathbb{R} \), with the usual metric.

1.1 Example

Each uniformly-continuous function \( f : (a, b) \to \mathbb{R} \), mapping a bounded open interval to \( \mathbb{R} \), is bounded. Indeed, given such an \( f \), choose \( \delta > 0 \) such that the modulus of continuity \( \omega_f(\delta) < 1 \), i.e.

\[ |x - y| < \delta \Rightarrow |f(x) - f(y)| < 1. \]

Take \( n \in \mathbb{N} \) greater than \((b-a)/\delta\), \( h = (b-a)/n \), and \( a_i = a + ih \) \((0 \leq i \leq n)\). Then

\[ |f(x)| \leq 1 + \max\{|f(a_i)| : 1 \leq i \leq n-1\}. \]

A very similar argument shows that if \( X \) is totally-bounded, then each uniformly-continuous function from \( X \) is bounded. However, this is not the whole story.

1.2 Example

Let \( X \) be the unit ball of \( \ell^\infty \), i.e. the space of all bounded sequences \( \{a_n\} \) of complex numbers., with the metric induced by the supremum norm:

\[ \rho(\{a_n\}, \{b_n\}) = \sup_n |a_n - b_n|. \]

Suppose \( f : X \to \mathbb{R} \) is uniformly-continuous, and choose \( \delta > 0 \) such that \( \omega_f(\delta) < 1 \). Let \( m \in \mathbb{N} \) be the ceiling of \( 1/\delta \). Then for each \( a = \{a_n\} \in X \), taking \( h = \sup_n |a_n|/m \) and \( b_i = iha \), we have

\[ |f(a)| \leq |f(0)| + \sum_{i=1}^m |f(b_i) - f(b_{i-1})| \leq |f(0)| + m. \]

Thus \( f \) is bounded. However, \( X \) is not totally-bounded.
2 Epsilon-step Territories

For \( \epsilon > 0 \) and \( a, b \in X \), we say that \( a \) is \( \epsilon \)-step-equivalent to \( b \) if there exist points \( a_0 = a, a_1, \ldots, a_n = b \), belonging to \( X \), with \( \rho(a_{i-1}, a_i) \leq \epsilon \) for each \( i \). This defines an equivalence relation on \( X \) (for each fixed \( \epsilon > 0 \)). We call the equivalence classes \( \epsilon \)-step territories, and denote the territory of a point \( a \) by \( T_\epsilon(a) \), or just \( T(a) \), if the value of \( \epsilon \) is clear from the context.

For \( \epsilon > 0 \) and \( a, b \in X \), we denote by \( s_\epsilon(a, b) \) (or just \( s(a, b) \)) the infimum of those \( n \in \mathbb{N} \) (if any) for which there exist \( a_0 = a, a_1, \ldots, a_n = b \) belonging to \( X \), with \( \rho(a_{i-1}, a_i) \leq \epsilon \). Obviously, \( s(a, b) < +\infty \) if and only if \( T(a) = T(b) \).

We say that a territory \( T(a) \) is \( \epsilon \)-step-bounded if

\[
\sup_{x \in T(a)} s(a, x) < +\infty,
\]

and we call this supremum the \( \epsilon \)-step extent of \( T(a) \).

We define a new ‘distance’ function on \( X \times X \), the \( \epsilon \)-step distance, by setting \( d_\epsilon(a, b) \) equal to

\[
\inf \left\{ \sum_{i=1}^{n} \rho(a_{i-1}, a_i) : a_0, a_1, \ldots, a_n \in X, a_0 = a, a_n = b, \text{ and } \rho(a_{i-1}, a_i) \leq \epsilon \right\},
\]

whenever \( a, b \in X \). This has all the properties of a metric, except that its value may be \( +\infty \). (One may obtain a proper metric by forming \( \arctan \circ d_\epsilon \).) The distance \( d_\epsilon \) is a proper metric when restricted to any particular \( \epsilon \)-step territory \( T(a) \). In general, \( d_\epsilon(a, b) \) is at least as large as the original \( \rho(a, b) \), but \( d_\epsilon(a, b) \) coincides with \( \rho(a, b) \) whenever \( \rho(a, b) \leq \epsilon \), and hence \( d_\epsilon \) induces the same topology as \( \rho \) on \( T \), and moreover a function \( f : X \to \mathbb{R} \) is \( \rho \)-uniformly-continuous if and only if it is \( d_\epsilon \)-uniformly-continuous. Indeed its \( \rho \)-modulus of continuity coincides with its \( d_\epsilon \)-modulus of continuity when the argument is less than or equal to \( \epsilon \).

One readily checks that a territory \( T \) is \( \epsilon \)-step-bounded if and only if its \( d_\epsilon \)-diameter is finite. Moreover, its \( \epsilon \)-step extent lies between

\[
\frac{d_\epsilon - \text{diam}(T)}{\epsilon} \quad \text{and} \quad 2 + \frac{2d_\epsilon - \text{diam}(T)}{\epsilon}.
\]

We now state the main result.
**Theorem 2.1** Let \((X, \rho)\) be a metric space. Then the following are equivalent:

1. Each uniformly-continuous function \(f : X \to \mathbb{R}\) is bounded.
2. For each \(\epsilon > 0\), \(X\) has only a finite number of \(\epsilon\)-step territories, and each territory is \(\epsilon\)-step-bounded.

**Proof.** (1) \(\Rightarrow\) (2): Suppose (1). Fix \(\epsilon > 0\).

Suppose that \(X\) has infinitely-many \(\epsilon\)-step-territories. Let \(T_n\) (for \(n = 1, 2, 3, \ldots\)) be distinct territories. Then the function \(f\), defined by

\[
f(x) = \begin{cases} 
  n & \text{if } x \in T_n, \\
  0 & \text{if } x \in X \sim \bigcup_{n=1}^{\infty} T_n,
\end{cases}
\]

is uniformly-continuous and unbounded, which is impossible. Thus \(X\) has only a finite number of \(\epsilon\)-step territories.

Now suppose that one of the \(\epsilon\)-step territories, say \(T(a)\), is not \(\epsilon\)-step-bounded. Define

\[
g(x) = \begin{cases} 
  d_\epsilon(a, x) & \text{if } x \in T(a), \\
  0 & \text{if } x \in X \sim T(a).
\end{cases}
\]

Then \(g\) is uniformly-continuous on \(X\), and unbounded, contradicting the assumption. Thus each \(\epsilon\)-step territory is \(\epsilon\)-step-bounded, and (2) holds.

(2) \(\Rightarrow\) (1): Suppose (2), and fix \(f : X \to \mathbb{R}\), uniformly-continuous.

Pick \(\delta > 0\) such that \(\omega_f(\delta) < 1\). With \(\epsilon = \delta\), choose \(a_1, \ldots, a_n \in X\) such that \(X = \bigcup_{j=1}^n T(a_j)\). Then take \(N\) to be the maximum of the \(\epsilon\)-step extents of the \(T(a_j)\), for \(1 \leq j \leq n\). Let \(M = \max_j |f(a_j)|\).

For each \(x \in X\), there exists \(j\) with \(x \in T(a_j)\), and then there are \(x_0 = a_j, x_1, \ldots, x_m = x\) belonging to \(X\), with \(m \leq N\) and \(\rho(x_{i-1}, x_i) \leq \epsilon\). Thus

\[
|f(x)| \leq m + |f(a_j)| \leq N + M.
\]

Thus \(f\) is bounded. This proves (1). \(\blacksquare\)