A TAUBERIAN THEOREM ARISING IN OPERATOR THEORY

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1. Introduction

The genesis of this paper is the observation that a recent result of Katznelson and Tzafriri [4] in operator theory uses a theorem of tauberian type (Theorem 2) that is very closely parallel to a tauberian theorem used by D. J. Newman [6] in his simple proof of the Prime Number Theorem. [See also Korevaar [5]. An adaptation of Newman's proof by Zagier, communicated privately by L. Zalcman, uses a third tauberian theorem of a similar kind.]

Katznelson and Tzafriri used arguments of harmonic analysis to prove their tauberian theorem. Our main result (Theorem 4) is a more general theorem of tauberian type which we prove by complex-variable methods which are an extension of Newman's approach. Apart from its operator-theoretic consequences (Theorem 5), this provides a new proof of the result of Katznelson-Tzafriri.

The simplest form of the Katznelson-Tzafriri result on operators is as follows. (We write $\Gamma = \{z \in \mathbf{C} : |z| = 1\}$).

THEOREM 1 ([4]). *Let X be a complex Banach space, $T \in L(X)$ and suppose that $\sup \|T^n\| < \infty$. Then $\|T^n - T^{n+1}\| \to 0$ if (and only if) $\mathrm{Sp}(T \cap \Gamma) \subseteq \{1\}$.*

In connection with this last result, we observe that the 'only if' part is trivial and that the condition that $\sup \|T^n\| < \infty$ immediately implies that $\mathrm{Sp}(T) \subseteq \{z \in \mathbf{C} : |z| \le 1\}$.

The special case of Theorem 1 in which $\mathrm{Sp}(T) = \{1\}$ was proved by Esterle [1] (Theorem 9.1) in a very brief and elegant argument, using a simple result about entire functions of minimal exponential type.

In [4], Katznelson and Tzafriri deduce their result from the following tauberian theorem.

THEOREM 2 ([4]). *Let (a_n) be a bounded sequence of complex numbers and set*

$$f(z) = \sum_{n \ge 0} a_n z^n \qquad (|z| < 1).$$

Suppose that every point of $\Gamma \setminus \{1\}$ is a regular point for f; then $a_n - a_{n+1} \to 0$ as $n \to \infty$.

Remark. We may not in general conclude that the sequence $(a_n - a_{n+1})$ is in an ℓ_p - class, as has been shown by T. J. Ransford in a private communication. In fact, if we take $(m_k)_{k\geq 1}$ to be a sequence of integers such that $m_1 > 1$, $m_{k+1} > 2m_k + 1$ $(k \geq 1)$, set $\theta_k = 1/\log m_k$ and then define rephrase ?

$$f(z) = \sum_{k=1}^{\infty} m_k^{\frac{1}{2}} \left[\frac{z(1 + \exp(i\theta_k).z)}{2} \right]^{2 m_k},$$

for all $z \in U = \{z \in \mathbf{C} : |z(1+z)/2| < 1\}$, then f is a well-defined analytic function on U. If $\sum a_n z^n$ is the Taylor series of f on $\{|z| < 1\}$, then $|a_n| \leq 1$ (all n), but $\sum |a_n - a_{n+1}|^p$ diverges for every $p \geq 1$. We shall not give the details.

To bring out the tauberian nature of Theorem 2, it is helpful to recast it in terms of the function $g(z) = (1-z)f(z)$. (We shall write Δ and Δ^- for the open and closed unit discs, respectively.)

THEOREM 2'. *Let $g(z)$ be analytic on $\Delta^- \setminus \{1\}$, with $g(z) = \sum_{k\geq 0} b_k z^k$ on Δ, where $\sup_N |\sum_0^N b_k| < \infty$. Then $\sum_{k\geq 0} b_k \zeta^k$ converges to $g(\zeta)$ for all $\zeta \in \Gamma \setminus \{1\}$.*

This last result is stronger than Theorem 2 in that it asserts the convergence of $\sum b_k \zeta^k$ on $\Gamma \setminus \{1\}$ and not merely that $b_k \to 0$. However, by a classical result of Fatou and M. Riesz (see e.g. [7], §7.31), if $\sum b_k z^k$ is a power series, with radius of convergence 1, such that $b_k \to 0$, then the series converges at all regular points of Γ. From Theorem 2' a more precise version of Theorem 1 follows (the notation being as in Theorem 1):

THEOREM 1'. *Let $T \in L(X)$ with $\sup \| T^n \| < \infty$ and such that $\mathrm{Sp}(T) \cap \Gamma \subseteq \{1\}$. Then, for every $\zeta \in \Gamma \setminus \{1\}$, the series $\sum_{n\geq 0} z^n (T^n - T^{n+1})$ is norm - convergent in $L(X)$.*

We observe that, as in [4], the deduction of the operator theorem from the result of complex analysis requires a form of Theorem 2 (or 2') in which the sequence (a_n) (respectively (b_k)) may consist of elements of a complex Banach space and not just complex numbers. This strengthening comes from entirely standard techniques of functional analysis, coupled with the observation that the convergence asserted in Theorem 2 (respectively Theorem 2') is uniform over certain normal families of functions. This fact comes with no extra effort from the given proofs and we shall make no further reference to the point, beyond a brief remark at the end of the proof of Theorem 4.

We shall now refer briefly to Newman's proof of the Prime Number Theorem, which was mentioned at the beginning of the paper. Newman's proof hinges on the following tauberian result, whose close analogy to Theorem 2' is evident.

THEOREM 3 (D. J. Newman [6]). *Let (a_n) be a bounded sequence of complex numbers and let*

$$f(z) = \sum_{n \geq 1} a_n n^{-z} \quad (\operatorname{Re} z > 1).$$

If every point of $\{\operatorname{Re} z = 1\}$ *is a regular point for f; then* $\sum_{n \geq 1} a_n n^{-1}$ *converges to* $f(1)$.

This tauberian theorem is, in fact, as is stated in [6], contained in a more general result of Ingham ([3]). In [6], Newman proves the theorem by an elegant and elementary use of the Cauchy integral formula. (For the Prime Number Theorem, the application is with $a_n = \mu(n)$, where μ is the Möbius function). In the next section, we shall adapt Newman's method to give a considerable extension of Theoem 2'.

2. A tauberian theorem.

THEOREM 4. *Let* $f(z) = \sum_{n \geq 0} a_n z^n$ *have radius of convergence* 1 *and let* $E \subset \Gamma$ *be its set of singular points. Suppose that E has linear measure zero and that, for some* $M > 0$,

$$\left| \sum_{n=0}^{N} a_n \zeta^n \right| \leq M \qquad (\text{all } N \geq 0, \text{ all } \zeta \in E).$$

Then $\sum_{n \geq 0} a_n z^n$ *converges to* $f(z)$ *for all z in* $\Gamma \setminus E$; *in particular*, $a_n \to 0$ *as* $n \to \infty$.

Remarks.

counter-example is provided by the binomial series $\sum_{n\geq 0} \binom{n+\alpha}{\alpha} z^n$, where $\alpha = -1 + ic$, with real $c \neq 0$. This series converges to the principal value of $(1 - z)^{ic}$ for all $z \in \Delta^-\setminus\{1\}$ but is not convergent at $z = 1$, where it oscillates finitely. In fact the series has the even stronger property that its partial sums are uniformly bounded on Δ^-; clearly 1 is the only singular point on Γ. The details of the calculations for this example may be found in [2], § 6.11.

2. We do not know whether the condition that E have linear measure zero is necessary for the conclusion that $a_n \to 0$ (and so for $\sum a_n z^n$ to converge on $\Gamma \setminus E$, by the Riesz result). One case at the opposite extreme from Theorem 4, where the conclusion does hold is that in which Γ is a natural boundary for the series, i.e. $E = \Gamma$. In that case, the conditions ensure that $f \in H^\infty$, so that $\sum a_n z^n$ converges for almost all $z \in \Gamma$, by Carleson's theorem, and certainly $a_n \to 0$.

Proof of Theorem 4. We may suppose that f is analytic on an open set $U \supset \Delta^-\setminus E$. By rotation, we may assume that $1 \in U$ and it is enough to prove convergence of the series at $z = 1$. (In view of the Riesz result, it would actually suffice to prove that $a_n \to 0$; however the direct proof of convergence of the series is hardly more difficult and we therefore give it, for the convenience of the reader.)

For most of the calculation it is convenient to change the variable, by putting $z = (w - i)/(w + i)$. Then, under the mapping $z \to w$, we have :

$\Delta \to \{ \operatorname{Im} w > 0 \}$;

$1 \to \infty$;

$E \to F$, say, a compact subset of $\mathbf{R}$ of linear measure zero;

$U \to V$, say, an open subset of $\mathbf{C}_\infty$ with

$V \supset \{ \operatorname{Im} w > 0 \} \cup \{\mathbf{R} \setminus F\} \cup \{\infty\}$.

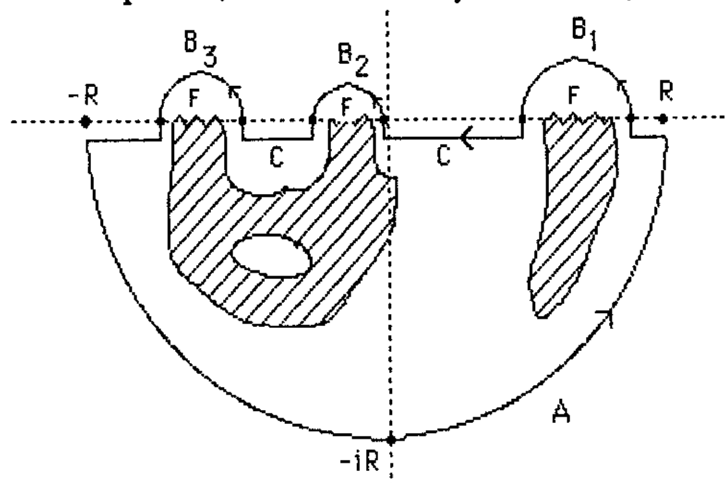
If $g(w) = \sum_{n\geq 0} a_n \left(\frac{w - i}{w + i}\right)^n$ then g is analytic on V and we have to prove that

$g_N(\infty) \to g(\infty)$ as $N \to \infty$, where we have written

$$g_N(w) = \sum_{n=0}^{N-1} a_n \left(\frac{w-i}{w+i}\right)^n .$$

Now choose $R > 2$ such that $V \supset \{w : |w| > R-2\}$ and then choose ε, $0 < \varepsilon < 1$.

We cover F by a finite collection of disjoint open intervals $I_1, \ldots, I_q$, where, say, $I_j = (\alpha_j - \varepsilon_j, \alpha_j + \varepsilon_j)$, such that $F \cap I_j \neq \emptyset$ (all j), $\sum_{j=1}^{q} \varepsilon_j \le \varepsilon$ and $1/2 \le \varepsilon_j / \varepsilon_k \le 2$ (all j,k). (To see that this choice is possible, recall that F is totally disconnected).



We now introduce a contour $\Gamma \subset V$, that is most easily described by reference to the figure: for $j = 1, \ldots, q$, we let B_j be the semicircle in the upper half-plane with diameter I_j, $B_j = \{ \alpha_j + \varepsilon_j e^{i\theta} : 0 \le \theta \le \pi \}$; C is the union of the $(q+1)$ subintervals of $[-R, R]$ that are complementary to $\bigcup_{j=1}^{q} B_j$, but translated a (small) distance $\delta > 0$ below the real axis, together with the 2q vertical segments, each of length δ, joining the ends of these latter subintervals to the ends of the semicircles B_j; the number δ is chosen so small that g is analytic on and above the curve segment formed by $C \cup \bigcup_{1}^{q} B_j$; the portion A of the contour is formed by that arc of the circle $\{|w| = R\}$ that is contained in the half-plane $\{\mathrm{Im}\, w \le -\delta\}$. We then form the simple closed path $\Gamma = A + C + \sum_{1}^{q} B_j$, by joining up in the obvious way; evidently $\Gamma \subset V$ and g is analytic on and outside Γ (including at ∞). In the figure, V is the unshaded region and we have taken $q = 3$.

Define the rational functions:

$$\theta(w) = \prod_{j=1}^{q} \left(1 - \frac{\varepsilon_j^2}{(w - \alpha_j)^2} \right), \quad \theta_N(w) = \left(\frac{w+i}{w-i} \right)^N \theta(w),$$

and then set

$$J_N = \int_\Gamma (g(w) - g_N(w))\,(w+i)^{-1}\theta_N(w)\,dw.$$

The integrand is analytic outside Γ (the singularity at $w = i$ is removable) and so, by Cauchy's theorem, for every $S > R$,

$$J_N = \int_{|w|=S} (g(w) - g_N(w))\,(w+i)^{-1}\,\theta_N(w)\,dw$$

$$= \int_{|w|=S} \big(g(\infty) - g_N(\infty)\big)\,(w+i)^{-1}\theta_N(w)\,dw \quad + \quad O\left(\frac{1}{S}\right),$$

as $S \to \infty$, so that

$$J_N = 2\pi i\,(g(\infty) - g_N(\infty)).$$

We remark that the above formula would, of course, also be valid without the factor $\theta_N(w)$, which satisfies $\theta_N(\infty) = 1$. However, this factor is chosen precisely to cancel the growth of the integrand near the singular set F. This device is in direct imitation of Newman's method in [6]; it is in this respect that our method is an extension of his.

To estimate J_N, we shall estimate separately:

(a) $$\int_{\Sigma B_j} \big(g(w) - g_N(w)\big)\,(w+i)^{-1}\,\theta_N(w)\,dw\ ;$$

(b) $$\int_{A+C} g_N(w)\,(w+i)^{-1}\,\theta_N(w)\,dw\ ;$$

(c) $$\int_{A+C} g(w)\,(w+i)^{-1}\,\theta_N(w)\,dw\,.$$

In carrying out these estimates we require the following technical lemma, whose proof we defer

LEMMA. *With the notation established above, let* $\sum b_n z^n$ *be a power series such that*

$$\left|\sum_{k=0}^{K} b_k \zeta^k\right| \le 2M \qquad (\textit{all } K \ge 0,\ \textit{all } \zeta \in E).$$

Then, for all $w \in \bigcup_{j=1}^{q} B_j$,

$$\left|\sum_{n\ge 0} b_n \left(\frac{w-i}{w+i}\right)^n (w+i)^{-1}\theta(w)\right| \le 16\,MR^2 .$$

We now complete the proof of Theorem 4 by carrying out the estimates for (a), (b) and (c) above.

(a) Let $w \in \bigcup_{j=1}^{q} B_j$. Then

$$\left(g(w) - g_N(w)\right)\left(\frac{w+i}{w-i}\right)^N = \sum_{n=0}^{\infty} a_{n+N}\left(\frac{w-i}{w+i}\right)^n .$$

If $\zeta \in E$ then, for all $K \ge 0$,

$$\left|\sum_{k=0}^{K} a_{k+N}\zeta^k\right| = \left|\sum_{p=N}^{N+K} a_p \zeta^p\right| \le 2M,$$

by the hypothesis on f. Hence, by the Lemma,

$$\left|\int_{\sum B_j} \left(g(w) - g_N(w)\right)(w+i)^{-1}\theta_N(w)\,dw\right| \le \sum_{j=1}^{q} \text{length}(B_j).16\,MR^2$$

$$= \sum_{j=1}^{q} \pi\varepsilon_j .16\,MR^2 \le 16\pi MR^2\varepsilon,$$

independently of N.

(b) The integrand $g_N(w)(w+i)^{-1}\theta_N(w)$ is analytic except at $\{i, \alpha_1, \alpha_2, \ldots, \alpha_q\}$. Hence, by Cauchy's Theorem, writing D_j for the reflection of B_j in the real axis $(j = 1, \ldots, q)$,

$$\int_{A+C} g_N(w)(w+i)^{-1}\theta_N(w)\,dw = -\int_{\sum D_j} g_N(w)(w+i)^{-1}\theta_N(w)\,dw .$$

An estimate very similar to that in (a) (considering the conjugate of the last integral, in order to have an integral over $\sum B_j$) then gives again,

$$\left|\int_{A+C} g_N(w)\,(w+i)^{-1}\,\theta_N(w)\,dw\right| \le 16\pi MR^2\varepsilon,$$

independently of N.

(c) Let $L = \sup_{w \in A+C} |g(w)\,(w+i)^{-1}\,\theta(w)|$, which is independent of N. Then,

$$\left|\int_{A+C} g(w)\,(w+i)^{-1}\,\theta_N(w)\,dw\right| \le L\int_{A+C} \left|\frac{w+i}{w-i}\right|^N |dw|.$$

Since $\left|\dfrac{w+i}{w-i}\right| < 1$ for all $w \in A+C$ (because $A+C \subset \{\operatorname{Im} w < 0\}$), it follows, by the Bounded Convergence Theorem, that there is some $N_0(\varepsilon)$ such that

$$L\int_{A+C} \left|\frac{w+i}{w-i}\right|^N |dw| < \varepsilon \qquad (N \ge N_0).$$

Combining the estimates for (a), (b) and (c), we have, for all $N \ge N_0$,

$$|2\pi i\,(g(\infty) - g_N(\infty))| \le (32\pi MR^2 + 1)\,\varepsilon.$$

Subject to the Lemma, this completes the proof that $g_N(\infty) \to g(\infty)$ as $N \to \infty$.

Proof of the lemma. Without loss of generality, suppose that $w \in B_1$, say $w = \alpha_1 + \varepsilon_1 e^{i\theta}$, where $0 \le \theta \le \pi$. The required inequality will result from the following four estimates:

(i) $\displaystyle\left|\sum_{n\ge 0} b_n \left(\frac{w-i}{w+i}\right)^n\right| \le \frac{8MR^2}{\sin\theta}$;

(ii) $|(w+i)^{-1}| \le 1$;

(iii) $|1 - \varepsilon_1^2/(w-\alpha_1)^2| = 2\sin\theta$;

(iv) $|1 - \varepsilon_j^2/(w-\alpha_j)^2| \le 1$, for all $j \ne 1$.

Of these, (ii) and (iii) are more or less immediate; we shall give some details for (i) and (iv).

(i) Let $\beta \in I_1 \cap F$ and write $z = (w-i)/(w+i)$, $\zeta = (\beta - i)/(\beta + i)$, so that $\zeta \in E$. Let

$$s_n = \sum_{k=0}^{n} b_k \zeta^k \quad (n \ge 0), \qquad s_{-1} = 0.$$

Then

$$\sum_{n\ge 0} b_n \left(\frac{w-i}{w+i}\right)^n = \sum_{n\ge 0} b_n z^n = \sum_{n\ge 0} b_n \zeta^n (z/\zeta)^n$$

$$= \sum_{n\ge 0} (s_n - s_{n-1})(z/\zeta)^n = \sum_{n\ge 0} s_n \left((z/\zeta)^n - (z/\zeta)^{n+1}\right).$$

By hypothesis, $|s_n| \le 2M$ for all n and so

$$\left|\sum_{n\ge 0} b_n \left(\frac{w-i}{w+i}\right)^n\right| \le 2M\,|1 - z/\zeta|\,\sum_{n\ge 0} |z/\zeta|^n$$

$$= 2M\,|\zeta - z| \,/\, (1 - |z|)$$

$$\le 4M\,|\zeta - z| \,/\, (1 - |z|^2).$$

It is now an elementary exercise to show that $|\zeta - z| \le 4\varepsilon_1$, while $1 - |z|^2 \ge 2\varepsilon_1 \sin\theta \,/R^2$; the estimate stated in (i) is now immediate.

(iv) Take $j \ne 1$; then

$$|1 - \varepsilon_j^2/(w - \alpha_j)^2| = |1 - \varepsilon_j^2/(\alpha_1 - \alpha_j + \varepsilon_1 e^{i\theta})^2|$$

will be less than or equal to 1 if and only if

$$|(\alpha_1 - \alpha_j + \varepsilon_1 e^{i\theta})^2/\varepsilon_j^2 - 1| \le |(\alpha_1 - \alpha_j + \varepsilon_1 e^{i\theta})^2/\varepsilon_j^2 - 0|,$$

which holds if and only if

$$\operatorname{Re}[(\alpha_1 - \alpha_j + \varepsilon_1 e^{i\theta})^2/\varepsilon_j^2] \ge 1/2.$$

But the validity of this last inequality is an elementary exercise, using the fact that $\varepsilon_j/\varepsilon_1 \ge 1/2$ for all $j \ne 1$. This completes the proof of the lemma.

Remark. It is plain from the proof of Theorem 4 that, for fixed U, E, M, if F is a normal family of functions analytic on U and satisfying the tauberian condition relative to E, M, then, for each $z \in \Gamma \backslash E$, the series $\sum a_n z^n$ converges to f(z) uniformly with respect to all f in F. By a standard Hahn - Banach argument, it follows that the theorem (with norm signs in appropriate places) is also valid for analytic functions taking their values in a complex Banach space. The following result, an extension of Theorem 1, is a deduction from Theorem 4 in this 'Banach-space valued' form. We remark that, in Theorem 5, the equivalence of (i) and (ii) could be deduced from results in [4], while (iii) is, we believe, new.

THEOREM 5. *Let* $\phi(z) = \sum c_n z^n$, *where* $\sum n \mid c_n \mid < \infty$. *Let T be a linear operator on a complex Banach space X, such that* $\sup \| T^n \| < \infty$. *Then the following statements are equivalent*:

(i) $\| \phi(T)\, T^n \| \to 0$ *as* $n \to \infty$;

(ii) $\mathrm{Sp}\ T \cap \Gamma \subset \{ \phi = 0 \}$;

(iii) $\mathrm{Sp}\ T \cap \Gamma$ *has linear measure zero, and* $\sum \phi(T)\, T^n z^n$ *converges to* $\phi(T)(I - zT)^{-1}$, *whenever* $z^{-1} \in \Gamma \backslash \mathrm{Sp}\ T$.

[*Remarks* (a) In this theorem $\phi(T)$ is *defined* as $\sum c_n T^n$, which is certainly norm-convergent in L(X), since $\sum n \mid c_n \mid < \infty$.

(b) The special case $\phi(z) = 1 - z$ gives Theorem 1.]

Proof. (iii) $\Rightarrow$ *(i)*: this is clear.

(i) $\Rightarrow$ *(ii)*: Let A be a maximal commutative subalgebra of L(X) that contains T and let Φ_A be its space of characters. Then

$$\mathrm{Sp}_{L(X)}(T) = \mathrm{Sp}_A(T) = \{ \chi(T) : \chi \in \Phi_A \}.$$

Let $\eta \in \mathrm{Sp}\, T \cap \Gamma$ and choose $\chi \in \Phi_A$ such that $\chi(T) = \eta$; then

$$|\phi(\eta)| = |\phi(\eta)\eta^n| = |\phi(\chi(T)).\, \chi(T)^n| = |\chi(\phi(T) T^n)| \le \|\phi(T) T^n\| \to 0,$$

as $n \to \infty$; thus $\phi(\eta) = 0$.

(ii) $\Rightarrow$ *(iii)*: Since $\phi \in A^+(T)$ (absolutely convergent Taylor series), condition (ii) certainly implies that $\mathrm{Sp}\, T \cap \Gamma$ has measure zero (except in the trivial case when $\phi = 0$). Let $f(z) = f(T)(I - zT)^{-1}$, which is analytic in a neighbourhood of $\bar{\Delta} \backslash E$, where $E = \{\zeta \in \Gamma : \zeta^{-1} \in \mathrm{Sp}\, T\}$. Since E is the conjugate of a zero-measure subset of Γ, it too has measure zero.

Now $f(z) = \sum \phi(T)\, T^n z^n \quad (|z| < 1)$, so that the result will follow from Theorem 4 (for Banach - space valued functions) provided that there is some positive constant M such that

$$\| \sum^{N} \phi(T) T^n \zeta^n \| \le M \quad \text{for all } N \ge 0 \text{ and all } \zeta \in E$$

To see that this last condition is in fact satisfied, note firstly that, writing $s_k(z) = \sum_{j=0}^{k} c_j z^j$, then for all $\zeta \in E$ we have :

$$\sum_{k=0}^{\infty}|s_k(\zeta^{-1})| = \sum_{k=0}^{\infty}|s_k(\zeta^{-1}) - \phi(\zeta^{-1})| = \sum_{k=0}^{\infty}\Big|\sum_{j=k+1}^{\infty} c_j(\zeta^{-1})^j\Big|$$

$$\leq \sum_{k=0}^{\infty}\sum_{j=k+1}^{\infty}|c_j| \leq \sum_{j=0}^{\infty} j|c_j| < \infty$$

Hence for all $\zeta \in E$ and $N \geq 0$,

$$\sum_{n=0}^{N}\phi(T)\,T^n\zeta^n = \sum_{n=0}^{N}\sum_{k=0}^{\infty} c_k\zeta^{-k}(\zeta T)^{n+k} = \sum_{n=0}^{N}\sum_{k=0}^{\infty}\big(s_k(\zeta^{-1}) - s_{k-1}(\zeta^{-1})\big)(\zeta T)^{n+k}$$

$$= \sum_{n=0}^{N}\sum_{k=0}^{\infty} s_k(\zeta^{-1})\big((\zeta T)^{n+k} - (\zeta T)^{n+k+1}\big) = \sum_{k=0}^{\infty} s_k(\zeta^{-1})\big((\zeta T)^{k} - (\zeta T)^{N+k+1}\big),$$

and so,

$$\Big\|\sum_{n=0}^{N}\phi(T)\,T^n\zeta^n\Big\| \leq \sum_{k=0}^{\infty}|s_k(\zeta^{-1})|\,\|(\zeta T)^k - (\zeta T)^{N+k+1}\| \leq \Big(\sum_{j=0}^{\infty} j|c_j|\Big)\,2\sup_n\|T^n\|,$$

which concludes the proof.

References

1. J. ESTERLE, 'Quasimultipliers, representations of H^∞, and the closed ideal problem for commutative Banach algebras', *Radical Banach algebras and Automatic Continuity*, (Lecture Notes in Mathematics, No. 975, Springer-Verlag, Berlin, Heidelberg, New York, 1983) 66 - 162.
2. G. H. HARDY, *Divergent Series,* OUP (1967 edition).
3. A. E. INGHAM, 'On Wiener's method in Tauberian theorems', *Proc. London Math. Soc.* (2) 38 (1935) 458 - 480.
4. Y. KATZNELSON & L. TZAFRIRI, 'On power-bounded operators', *J. Functional Analysis,* 68 (1986) 313 - 328.
5. J. KOREVAAR, 'On Newman's Quick Way to the Prime Number Theorem', *Math. Intelligencer,* 4 (1982) 108 - 115.

6. D. J. NEWMAN, 'Simple Analytic Proof of the Prime Number Theorem', *Amer. Math. Monthly,* 87 (1980) 693 - 696.

7. E.C. TITCHMARSH, *The Theory of Functions,* OUP (1968 edition).

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