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Rational Approximation and Weak Analyticity. II

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1. Introduction

(1.1) This paper is about rational approximation in Lipα norm $(0<\alpha<1)$ on compact sets $X\subset\mathbf{C}$. It contains Lipα analogues of uniform approximation theorems.

Fix $\alpha\in(0,1)$, and a compact set $X\subset\mathbf{C}$. Let $\beta=1+\alpha$. We denote by $R_\alpha(X)$ the closure in $\mathrm{Lip}(\alpha,X)$ of the set of rationals having poles off X, or, what comes to the same thing, of the set of functions holomorphic on a neighbourhood of X. In [OF 1] we showed that a function $f\in\mathrm{lip}(\alpha,\mathbf{C})$ belongs to $R_\alpha(X)$ if and only if either of the following two equivalent conditions holds:

(1) There exists $\kappa>0$ such that

$$\left|\int f\frac{\partial\phi}{\partial\bar{z}}dxdy\right|\leqq\kappa\cdot\|f\|_{\alpha,D}\cdot d\cdot\|\nabla\phi\|_\infty\cdot M^\beta(D\sim X),$$

whenever ϕ is a test function (i.e. C^∞ with compact support), sptϕ is a subset of a disc$D$, and $d=\mathrm{diam}D$.

(2) There exists $\eta(\delta)\downarrow 0$ as $\delta\downarrow 0$ such that

$$\left|\int f\frac{\partial\phi}{\partial\bar{z}}dxdy\right|\leqq\eta(d)\cdot d\cdot\|\nabla\phi\|_\infty\cdot M^\beta(D\sim X),$$

whenever ϕ is a test function, sptϕ is contained in a disc$D$, and $d=\mathrm{diam}D$.

Here, M^β denotes β-dimensional Hausdorff content.

These conditions express a kind of "weak analyticity" of the function f. The main objective of this paper is to examine the connection between $R_\alpha(X)$ and stronger kinds of "weak analyticity".

Our first result is a kind of Morera theorem.

Theorem 1. *Let* $0<\alpha<1$ *and let* $f\in\mathrm{lip}(\alpha,\mathbf{C})$. *Then* f *belongs to* $R_\alpha(X)$ *if and only if either of the following equivalent conditions hold:*

(3) *There exists* $\kappa>0$ *such that for each closed square* B,

$$\left|\int_{\mathrm{bdy}B}f(\zeta)d\zeta\right|\leqq\kappa\cdot\|f\|_{\alpha,B}\cdot M^\beta(B\sim X).$$

(4) *There exists* $\eta(\delta)\downarrow 0$ *such that for each closed square* B,

$$\left|\int_{\mathrm{bdy}\,B} f(\zeta)d\zeta\right| \leq \eta(\mathrm{side}\,B)\cdot M^{\beta}(B\sim X).$$

This theorem is the Lipα analogue of [V, Theorem 2, p. 171]. The idea of our proof could be used to give a somewhat novel proof of Vitushkin's result. We exploit the observation that the distributional $\bar\partial$ derivative $\frac{\partial\chi_E}{\partial\bar z}$ of the characteristic function χ_E of a smoothly bounded set E is the measure $-\left.\frac{dz}{2i}\right|_{\mathrm{bdy}\,E}$.

There is a "fine topology" on the plane which is important for Lipα approximation, the β-topology (it has nothing to do with Stone-Ček; β is $1+\alpha$). Let $B(a,r)$ denote the closed disc having centre a and radius r. We say that a set $N\subset\mathbf{C}$ is a *β-neighbourhood* of a if $a\in N$ and

$$\lim_{r\downarrow 0}\frac{M^{\beta}(B(a,r)\sim N)}{r^{\beta}}=0,$$

i.e. if $\mathbf{C}\sim N$ has β-density zero at a. That this does define a topology follows from the subadditivity of M^{β}. The 2-topology is the Lebesgue density topology.

We denote by $W^{1,p}_{\mathrm{loc}}$ the Sobolev space of functions $f\in L^p_{\mathrm{loc}}$ whose distributional derivative is also locally p-th power integrable.

Theorem 2. *Let* $0<\alpha<1$, $\beta=1+\alpha$, $p>2/(1-\alpha)$, *and* $f\in W^{1,p}_{\mathrm{loc}}$. *Then* $f\in R_\alpha(X)$ *if and only if* $\frac{\partial f}{\partial\bar z}=0$ *a.e.* $(dxdy)$ *on the β-interior of* X.

This parallels Theorem 1 of [OF 2]. The β-interior has replaced the set of nonpeak points. A weaker result was given in [OF 3, Theorem 2].

The instability theorem for the pair (*area*, M^{β}) [OF 4] yields that at area-almost-all $a\in\mathbf{C}$, either

$$\limsup_{r\downarrow 0}\frac{M^{\beta}(B(a,r)\sim X)}{r^{\beta}}>0 \quad\text{or}\quad \lim_{r\downarrow 0}\frac{M^{\beta}(B(a,r)\sim X)}{r^{2}}=0.$$

This permits us to strengthen Theorem 2, as follows:

Theorem 2′. *Let* α, β, p, *and* f *be as in Theorem 2. Then* $f\in R_\alpha(X)$ *if and only if* $\frac{\partial f}{\partial\bar z}=0$ *at dxdy-almost all points at which*

$$\lim_{r\downarrow 0}\frac{M^{\beta}(B(a,r)\sim X)}{r^{2}}=0.$$

In connection with [OF 2], we observe that, even for Lip1 functions f and int$X$ dense in X, the conditions $f\in R(X)$ and $f\in R_\alpha(X)$ are not equivalent. Take, for example, a compact set $E\subset B(0,1)$, having positive area and no interior. Take open, smoothly-bounded sets $\Omega_n\downarrow E$. For each n, take an open neighbourhood U_n of bdy(Ω_n), disjoint from E, with

$$M^{\beta}(U_n)<2^{-n}\cdot\mathrm{dist}(U_n,E)^{10}.$$

Let $U=\{U\}_{n=1}^{\infty}$, and let $X=B(0,2)\sim U$. Then for each $a\in E$, we obtain

$$(5)\qquad \liminf_{r\downarrow 0}\frac{\gamma(B(a,r)\sim X)}{r}>0.$$

$$(6)\qquad \lim_{r\downarrow 0}\frac{M^{\beta}(B(a,r)\sim X)}{r^2}=0.$$

By Nguyen's theorem [N] E may be chosen so that there is a Lip 1 function f, holomorphic off E, with $\frac{\partial f}{\partial \bar z}\neq 0$ on a set $E_1\subset E$ having positive area. By [OF 2, Theorem 1], $f\in R(X)$, whereas by Theorem 2, $f\notin R_\alpha(X)$.

Perhaps it is worth mentioning that Theorem 2 permits another equivalent characterisation of finely holomorphic functions. It is known [L] that a function f is finely holomorphic on the finely open set U if and only if each $a\in U$ has a Euclidean-compact fine neighbourhood X such that $f\in R(X)$. Combining Theorem 2 with Lyon's method [L], we can show that, given $\alpha\in(0,1)$, f is finely holomorphic on U if and only if each $a\in U$ has a Euclidean-compact fine neighbourhood X such that $f\in R_\alpha(X)$. This, however, is not very surprising. Fuglede has essentially shown [F, Theorem 11(a)] that, given $k\in Z$, the function f is finely holomorphic on U if and only if each $a\in U$ has a Euclidean-compact fine neighbourhood X such that f belongs to the closure of the rationals in $C^k(X)$.

2. Proof of Theorem 1

(2.1) Evidently, (3) $\Rightarrow$ (4). That $f\in R_\alpha(X)$ implies (3) is basically due to Dolženko [D] (cf. also [G, p. 65, Lemma 2.2]). For $f\in R_\alpha(X)$ and test functions ϕ, the integral $\int f\frac{\partial\phi}{\partial\bar z}dxdy$ is the coefficient $a_1(T_\phi f)$ in the expansion $T_\phi f(z)=a_1/z+a_2/z^2+\dots$, where $T_\phi f=\phi f-\mathscr{C}\left(f\frac{\partial\phi}{\partial\bar z}\right)$ and $\mathscr{C}$ denotes the *Cauchy transform*, defined by

$$\mathscr{C}g(z)=\frac{1}{\pi}\int\frac{g(u+\imath v)dudv}{z-(u+\imath v)}.$$

Since $\|T_\phi f\|\leq\kappa\cdot d(\phi)\cdot\|\nabla\phi\|_\infty\cdot\|f\|_{\alpha,D(\phi)}$ [OF 1], the result (3) follows from the fact that $T_\phi f$ is analytic off $D(\phi)\sim X$ and the Dolženko estimate

$$|a_1(g)|\leq\kappa\cdot\|g\|_\alpha\cdot M^\beta(\text{sing. spt}\,g).$$

It remains to show that (4) implies that $f\in R_\alpha(X)$.

For a square D, and $s>0$, let sD denote the square, concentric with and parallel to D, with side equal to s times the side of D.

(2.2) Lemma. *Suppose $f\in\operatorname{lip}(\alpha,X)$ and there exists $\eta(\delta)\downarrow 0$ as $\delta\downarrow 0$ such that*

$$\left|\int f\frac{\partial\phi}{\partial\bar z}dxdy\right|\leq\eta(d)\cdot d\cdot\|\nabla\phi\|_\infty\cdot M^\beta(D\sim X),$$

whenever $\phi\in\operatorname{Lip}1$, ϕ is supported on a square D of side d, and ϕ is constant on each square bdy(sD). *Then $f\in R_\alpha(X)$.*

Proof. This Lemma is a variant, with weakened hypotheses, of the implication (4) $\Rightarrow$ (1) in Theorem 1, p. 104 of [OF 1]. To prove it, look at the argument of pp. 199–202 of [OF 8], the lipα version of the Vitushkin scheme. You will see that to prove the Lemma, it suffices to produce, for each $\delta > 0$, a partition of unity $\{\phi_n\} = \{\phi_n^\delta\}$ such that

(7) $$0 \leqq \phi_n \leqq 1\,, \quad \phi_n \in \mathrm{Lip}\,1\,, \quad \|\nabla\phi_n\|_\infty \leqq \sqrt{2}/\delta\,,$$

ϕ_n is supported on a square D_n of side δ, no point belongs to more than 9 of the D_n, ϕ_n is constant on each square $\mathrm{bdy}(tD_n)$,

and such that, given a positive number $\beta = \beta_n < \delta$, we may write $\phi_n = \sum_k \psi_k$ (with the ψ_k depending on β_n), where

(8) $$0 \leqq \psi_k \leqq 1\,, \quad \psi_k \in \mathrm{Lip}\,1\,, \quad \|\nabla\psi\|_\infty \leqq \sqrt{2}/\beta\,,$$

ψ_k is supported on a square E_k of side β, no point belongs to more than 9 of the E_k, ψ_k is constant on each square $\mathrm{bdy}(tE_k)$.

The "variable-size" subpartition $\psi_k^{(n)}$ is used for matching the second coefficient in the Laurent expansion at infinity of $T_{\varphi_n} f$, where T_φ is the Vitushkin localisation operator.

The whole problem here is to find functions φ_n and ψ_k that are *level on squares*.

(2.3). Construction of the Special Partition of Identity. Let D_a denote the open square of side 2 centred at the point $a \in \mathbf{C}$ with sides parallel to the axes and let E_a denote the square obtained from D_a by rotating through $\pi/4$ and shrinking by $1/\sqrt{2}$, keeping the centre at a. Figure 1 shows E_0 inside D_0.

Define

$$\varrho(x, y) = \max\{0, 1 - \max\{|x|, |y|\}\}\,,$$
$$\sigma(x, y) = \max\{0, 1 - \max\{|x - y|, |x + y|\}\}\,.$$

Then ϱ (resp., σ) has support D_0 (resp., E_0), is constant on each $\mathrm{bdy}(tD_0)$ (resp., $\mathrm{bdy}(tE_0)$), has values in $[0, 1]$, and is Lip 1, with $\|\nabla\varrho\|_\infty \leqq 1$ (resp., $\|\nabla\sigma\|_\infty \leqq \sqrt{2}$). For $(m, n) \in Z^2$, define

$$\psi_{m,n}(x, y) = \begin{cases} \varrho(x - m, y - n)\,, & \text{if } m + n \text{ is even}\,, \\ \sigma(x - m, y - n)\,, & \text{if } m + n \text{ is odd}\,. \end{cases}$$

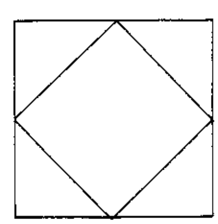

Fig. 1

Then $\{\psi_{m,n}:(m,n)\in Z^2\}$ is a partition of unity. Indeed, it suffices to check this on the triangle with vertices $(0,0)$, $(1,0)$, $(1,1)$, where we obtain

$$\sum_{m,n} \psi_{m,n} = \psi_{0,0} + \psi_{1,0} + \psi_{1,1} = 1 - x + y + x - y = 1 .$$

Each $\psi_{m,n}$ is supported on a D_a or an E_a and is level on the corresponding $\mathrm{bdy}(tD_a)$ or $\mathrm{bdy}(tE_a)$.

The partition $\psi_{m,n}$ may be understood geometrically in terms of a simple tessellation of the layer $\{(x,y,z)\in R^3: 0 \leqq z \leqq 1\}$. The $\varrho(x-m, y-n)$ with *both* m and n even correspond to copies of Cheop's pyramid sitting edge to edge, covering the (x,y)-plane. The $\varrho(x-m, y-n)$ with both m and n odd correspond to identical pyramids, placed upside-down, with their vertices at the points where the first lot meet in fours. The functions $\sigma(x-m, y-n)$ with $m+n$ odd then describe the vertical thickness of the "holes" in the paving by pyramids. The holes are tetrahedra, which, viewed from above, appear as in Fig. 2.

If you ever find yourself in Paris, then for the price of a cup of coffee, you can see precisely such a tessellation in the cafeteria at Jussieu.

This geometrical approach makes it easy to see that, given any integer q, the pyramids corresponding to $\psi_{m,n}$ are exactly decomposable into similar pyramids and tetrahedral, reduced in scale by a factor q. The resulting algebraic formula, which can, of course, be checked algebraically, is the following: Let $\psi^{\delta}_{m,n} = \psi_{m,n}(z/\delta)$ be the partition $\psi_{m,n}$, rescaled by δ. Then for $0 < q \in Z$ and $\delta > 0$, we get

$$\phi^{\delta}_{m,n}(z) = \sum_{r,s} \psi^{\delta}_{m,n}(c_{rs}) \cdot \psi^{\beta}_{r,s}(z), \tag{9}$$

where $\beta = \delta/q$ and $c_{rs} = (r\beta, s\beta)$ is the centre of the support of $\psi^{\beta}_{r,s}$.

Evidently, the partition $\phi_n = \psi^{\delta}_{m,n}$ has property (7), and the decomposition (9) gives ψ_k, with property (8).

(2.4) Now we will prove that (4) implies the hypothesis of Lemma (2.2). This will complete the proof of Theorem 1.

Suppose $f \in \mathrm{Lip}(\alpha, X)$ and satisfies (4). Fix a square D of side $d > 0$ and suppose $\varphi \in \mathrm{Lip}\,1$ is constant on each $S_t = \mathrm{bdy}(tD)$ and zero off D. Then φ may be written

$$\varphi(z) = \varphi(a) + \int_0^1 \chi_{tD}(z)\psi(t)\,dt ,$$

where χ_{tD} is the characteristic function of tD, $\psi(t) = -\dfrac{d}{dt}\varphi(a + d \cdot t)$, and a is the centre of D. Formally, this suggests the formula

$$\int f \frac{\partial \varphi}{\partial \bar{z}}\, dx\,dy = \int_0^1 \left(\frac{1}{2i} \int_{S_t} f(z)\,dz \right) \psi(t)\,dt .$$

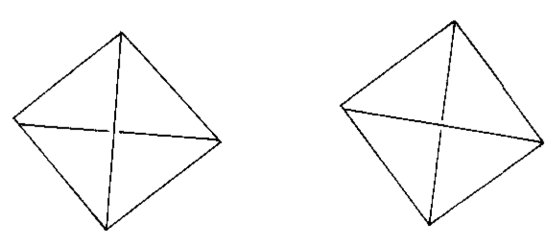

Fig. 2

In fact, this formula is readily verified, using Fubini's theorem.

We deduce that

$$\left|\int f\frac{\partial\varphi}{\partial\bar z}dxdy\right| \leqq \frac{1}{2}\cdot\|\psi\|_\infty\cdot\sup_{0\leqq t\leqq d}\left|\int_{S_t} f(z)dz\right|$$
$$\leqq \frac{1}{2}\cdot d\cdot\|\nabla\varphi\|_\infty\cdot\eta(2d)\cdot M^\beta(D\sim X),$$

which yields the desired estimate.

3. Proof of Theorem 2

(3.1) Necessity. For this we need only $f\in W^{1,1}$. Suppose $f\in R_\alpha(X)$. By Theorem 1, there exists $\kappa>0$ such that

$$\left|\int_{\mathrm{bdy}\,B} f(\zeta)d\zeta\right| \leqq \kappa\cdot\|f\|_{\alpha,B}\cdot M^\beta(B\sim X)$$

for each square B. Now

$$\int_{\mathrm{bdy}\,B} f(\zeta)d\zeta = 2i\int_B \frac{\partial f}{\partial\bar z}dxdy.$$

Let a be a point of the Lebesgue set of $\frac{\partial f}{\partial\bar z}$, at which

$$\frac{M^\beta(B(a,r)\sim X)}{r^2}\to 0$$

as $r\downarrow 0$. By [OF 4], area-almost-all points a of the β-interior of X are of this kind. Then

$$\frac{\partial f}{\partial\bar z}(a)\leftarrow\frac{1}{4s^2}\int_{B(a,s)}\frac{\partial f}{\partial\bar z}dxdy\to 0$$

as $s\downarrow 0$, so that $\frac{\partial f}{\partial\bar z}(a)=0$. This proves that the condition is necessary.

(3.2) Sufficiency. Recall [OF 6] that if $T\in\mathrm{Lip}(\alpha,X)^*$ annihilates the constants, then the Cauchy transform $\mathscr{C}f$ is representable by integration against a function (also denoted $\mathscr{C}f$) in L^q_{loc}, where q is the conjugate index to p [assuming that $p>2/(1-\alpha)$, of course]. In fact, if μ is a positive measure on $X\times X$, having no mass on the diagonal, and such that

$$Tg=\int_{X\times X}\frac{g(z)-g(w)}{|z-w|^\alpha}d\mu(z,w)$$

(– the De Leeuw representation), then

$$\mathscr{C}T(a)=\int_{X\times X}\frac{(w-z)d\mu(z,w)}{(z-a)(w-a)|z-w|^\alpha}$$

for $dxdy$-almost all a, and at almost all a we have

$$\int_{X \times X} \frac{|w-z|^{1-\alpha} d\mu(z,w)}{|z-a|\,|w-a|} < +\infty . \tag{10}$$

Suppose $T \perp R_\alpha(X)$ and μ is as above. We have

$$Tf = \int \mathscr{C}T \cdot \frac{\partial f}{\partial \bar{z}}\, dxdy ,$$

so it suffices to show that $\mathscr{C}T = 0$ at $dxdy$-almost all points off the β-interior of X.

Fix a point $a \notin \beta\text{-int}(X)$ at which (10) holds. There exist $\kappa > 0$ and $r_n \downarrow 0$ such that

$$\frac{M^\beta(U(a,r_n) \sim X)}{r_n^\beta} > \kappa > 0 ,$$

where $U(a,r)$ denotes the open disc with centre a and radius r. By [OF 7], there is a function $h(r) > 0$ with $h(r) \leqq r^\beta$ and $h(r)/r^\beta \to 0$, and a measure ν_n on $U(a,r_n) \sim X$ with $\nu B(b,r) \leqq r^\beta$ for all $b \in \mathbf{C}$ and all $r > 0$, and with $\|\nu_n\| \geqq \kappa r_n^\beta$. It follows that the Cauchy transform $\mathscr{C}\nu_n$ belongs to $R_\alpha(X)$, so that $T(\mathscr{C}\nu_n) = 0$. Moreover, $\|\nu_n\|^{-1} \cdot \mathscr{C}\nu_n \to (z-a)^{-1}$ pointwise off $\{a\}$, and an estimate shows that

$$\|\nu_n\|^{-1} \cdot |\mathscr{C}\nu_n(z) - \mathscr{C}\nu_n(w)| \leqq \kappa' \frac{|z-w|}{|z-a|\,|w-a|} .$$

By dominated convergence, $T\mathscr{C}\nu_n \to \mathscr{C}T(a)$, whence $\mathscr{C}T(a) = 0$. That does it.

In closing, we remark that the condition $p > 2/(1-\alpha)$ is a natural restriction, since without it a function in $W^{1,p}$ need not belong to Lipα.

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