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Inventiones mathematicae

C^∞ maps may increase C^∞-Dimension

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1. In Gromov's book [1, Sect. (1.3.2)] the C^∞-dimension of an arbitrary subset S in a smooth manifold Y is defined to be the least integer m such that S is contained in a countable union of C^∞-submanifolds of dimension m in Y.

Gromov observed that the Thom equisingularity theorem shows that the C^∞-dimension is monotone nonincreasing under *generic* C^∞ maps. He conjectured that this fails for arbitrary C^∞ maps. The purpose of this note is to give an example. Specifically, there is a C^∞ function $\chi: \mathbb{R} \to \mathbb{R}^2$ such that $\operatorname{im}\chi$ cannot (even) be covered by a countable union of nonsingular C^1 curves.

2. We appeal to the following simple fact, which is probably well-known.

Lemma. *If A and B are uncountable compact subsets of $\mathbb{R}$, then $A \times B \subset \mathbb{R}^2$ cannot be covered by a countable family of nonsingular C^1 curves.*

Proof. Suppose $\mathscr{F}$ is such a family. Replacing, if need be, each curve in $\mathscr{F}$ by a countable number of subcurves, we may write $\mathscr{F} = \mathscr{F}_1 \cup \mathscr{F}_2$ where no curve in $\mathscr{F}_1$ has a vertical tangent, and no curve in $\mathscr{F}_2$ has a horizontal tangent.

Take probability measures μ on A and ν on B having no point masses. The set $\cup\mathscr{F}_1$ meets each vertical line in a countable set, so by Fubini's theorem,

$$(\mu \times \nu)(\cup\mathscr{F}_1) = 0.$$

Similarly, $(\mu \times \nu)(\cup\mathscr{F}_1) = 0$, hence $(\mu \times \nu)(A \times B) = 0$, which contradicts $(\mu \times \nu)(A \times B) = (\mu A)(\nu B) = 1$.

We will construct a C^∞ function $\chi: \mathbb{R} \to \mathbb{R}^2$ for which $\chi(\{\chi' = 0\})$ (= the image of the set of critical points of χ) is $A \times A$ for a certain nonempty perfect set A. This will do.

3. For $2 \leq n \in Z$ and a sequence $d = \{d_j\}_0^\infty$ with $n d_{j+1} < d_j$, let $C(n, d)$ denote the linear Cantor set $\bigcap_{j=0}^{\infty} C_j$, obtained by starting with $C_0 = [0, d_0]$ and, at the j^{th} stage, replacing each of the n^{j-1} intervals in C_{j-1} by n equal, equally-spaced, maximally-spaced subintervals, each of length d_j.

Let $D = C(4, 16^{-j})$. (The important thing about 16 in this construction is that it exceeds 4.)

We will produce a sequence d_j and a map $\psi: \mathbb{R} \to \mathbb{R}$ such that ψ is C^∞, D is the set of critical points of ψ,

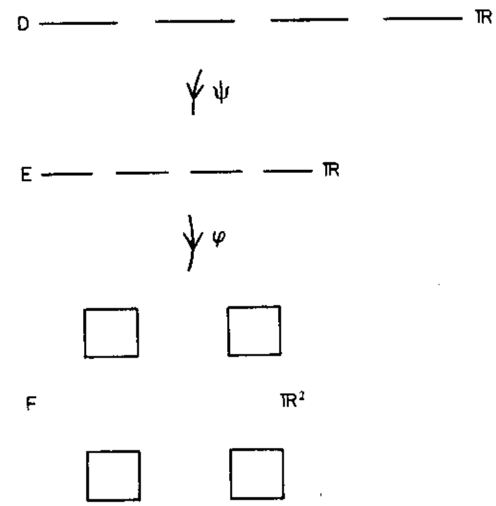

Fig. 1

and $\psi(D)$ is a Cantor set

$$E = C(4, d_j).$$

Then we will define a Lipschitzian (=Lip 1) map $\phi: \mathbb{R} \to \mathbb{R}^2$ such that ϕ is affine and nonconstant on each interval of $\mathbb{R} \sim E$, and $\phi(E)$ is the *square* Cantor set

$$F = C(2, d) \times C(2, d)$$

corresponding to the same sequence $d = \{d_j\}_0^\infty$.

The function $\chi = \phi \circ \psi$ then provides the example:

(1) The image of D under χ is F.

(2) χ is C^∞, and flat on D (i.e. $\chi^{(k)} = 0$ on D for $k = 1, 2, 3, \ldots$). To see this it is enough to check that $\chi^{(k)}(y) \to 0$ as y approaches any point $x \in D$ from $\mathbb{R} \sim D$. But for $y \in \mathbb{R} \sim D$ we have

$$\chi^{(k)}(y) = \psi^{(k)}(y) \cdot [\phi'(\psi(y))]^k,$$

since ϕ is affine on $\mathbb{R} \sim E$. Furthermore,

$$|\phi'(\psi(y))| \leqq L$$

where L is the Lipschitz constant of ϕ, so

$$|\chi^{(k)}(y)| \leqq |\psi^{(k)}(y)| \cdot L^k \to 0$$

as y approaches $x \in D$, since $\psi^{(k)}$ is continuous and vanishes at x.

4. To construct ψ and $\{d_j\}$, take a nonnegative C^∞ function λ that vanishes precisely on D, and set

$$\psi(x)=\int_0^x \lambda(t)\,dt.$$

To be specific, we define $\lambda(t)=0$ for $t\in D$ and

$$\lambda(t)=\exp\{-|t-a|^{-1}-|t-b|^{-1}\}$$

for $t\in(a, b)$, where (a, b) is any one of the bounded component intervals of $\mathbb{R}\sim\mathbb{D}$. It will be important that λ has the same integral over components of $\mathbb{R}\sim D$ having equal length. In does not much matter how we define λ off $[0, 1]$. For instance, $\lambda(t)=\exp\{-|t|^{-1}\}$ on $(-\infty, 0)$ and $\lambda(t)=\exp\{-|t-1|^{-1}\}$ on $(1, \infty)$ will be fine.

The function ψ is C^∞, and its critical points are precisely the points of D. The fact that $\psi(b)-\psi(a)=\psi(b')-\psi(a')$, whenever (a, b) and (a', b') are components of $\mathbb{R}\sim D$ having equal length, ensures that $E=\psi(D)$ is a Cantor set $C(4, d)$, and that defines the sequence $d=\{d_j\}$.

We can calculate d_j explicitly. It is just $\int_0^{16^{-j}} \lambda(t)\,dt$. Writing this out explicitly as a sum, it becomes

$$3\cdot\sum_{k=1}^{\infty} 4^k \int_0^{16^{-j-k}} \exp\{-t^{-1}-|16^{-j-k}-t|^{-1}\}\,dt.$$

From this, we obtain that

$$d_j\leqq\exp(-16^j)\cdot d_{j-1}.$$

Thus $d_j\to 0$, very rapidly. We will use only that $16d_j<d_{j-1}$ in the sequel.

5. To construct the map $\phi:\mathbb{R}\to\mathbb{R}^2$, we begin by constructing its restriction to E.

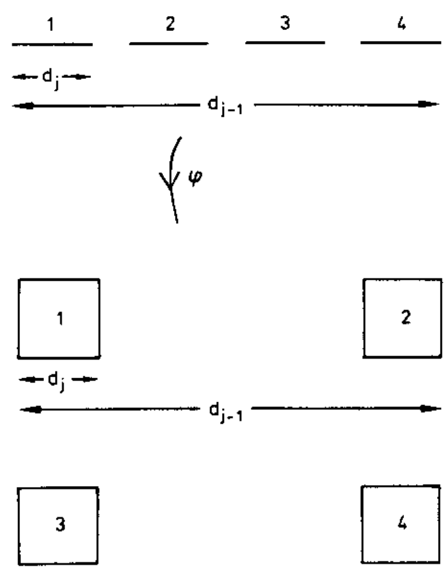

Fig. 2

The definition of ϕ is essentially explained by the diagram. Formally, it is the pointwise limit on E of maps ϕ_j. The map ϕ_j maps each interval composing $C_j = C_j(4, d)$ to the centre of one of the squares composing $C_j(2, d) \times C_j(2, d)$. The map preserves "order", where the "order" on E is the order of $\mathbb{R}$, and the "order" on $F = C(2, d) \times C(2, d)$ is that induced by top-left, top-right, bottom-left, bottom-right on j-th generation subsquares of a given $(j-1)$st. generation square.

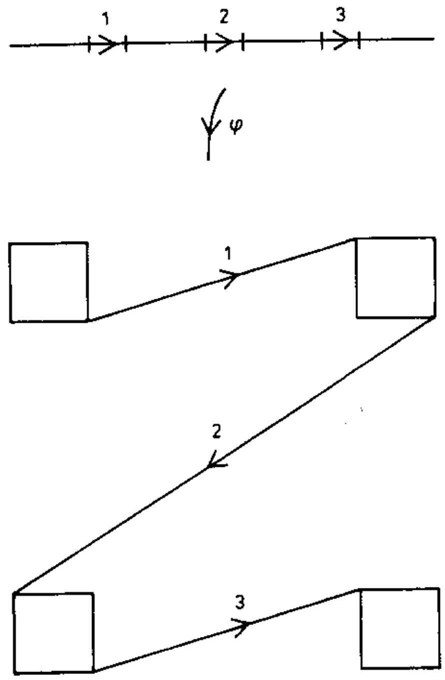

Fig. 3

To define ϕ on $\mathbb{R} \sim E$, we use the pattern indicated by the second diagram. Thus, ϕ is affine on each interval composing $\mathbb{R} \sim E$, and $\operatorname{im} \phi$ connects the bottom right corner of some square to the top left corner of "the next" square.

We have to check that ϕ is Lipschitzian.

1° First, it is Lip 1 on E. Fix $x, y \in E$, $x \neq y$. Pick j minimal so that x and y fall in different segments of $C_j(4, d)$. Since $16 d_j < d_{j-1}$, we see that $|x-y|$ is between $\frac{1}{4} d_{j-1}$ and d_{j-1}, and $|\phi(x) - \phi(y)|$ is between $\frac{1}{2} d_{j-1}$ and d_{j-1}, whence

$$|\phi(x) - \phi(y)| \leqq 4 |x-y|.$$

2° Next, the derivative ϕ' is bounded on $\mathbb{R} \sim E$. In fact, if we consider a segment of $\mathbb{R} \sim E$, that appears at the j^{th} generation, then both it and its image have lengths comparable to d_{j-1}.

3° Finally, given $x \in E$ and $y \in \mathbb{R} \sim E$, let z be the endpoint of y's segment that is nearest x. Obviously,

$$\begin{aligned}|\phi(x)-\phi(y)| &\leqq |\phi(x)-\phi(z)|+|\phi(z)-\phi(y)| \\ &\leqq 4|x-z|+(\sup_{\mathbb{R}\sim E}|\phi'|)\,|z-y| \\ &\leqq A|x-y|,\end{aligned}$$

where $A=\max\{4, \sup_{\mathbb{R}\sim E}|\phi'|\}$.

(In fact, closer analysis shows ϕ is biLipschitzian, for what it's worth.) This completes the construction of the example.

6. Addendum

A related question is whether there is a reasonable dimension concept "dim", with the following properties:

(1) $\dim E$ is defined for all subsets (or even closed subsets) of each C^∞ manifold, and $E\subset F \Rightarrow \dim E \leqq \dim F$.

(2) $\dim \mathbb{R}^m = m$.

(3) $\dim f(E) \leqq \dim E$ whenever f is a C^∞ map.

(4) There is a "topological Sard theorem": If f is C^∞ and maps a closed set A into B and $\dim B \geqq \dim A$, then the image of the critical set of $f\,|\,A$ is meagre in B.

This question was raised by Gromov, in reaction to the example. Basically the answer is no.

For properties (1) and (3) to make sense, the values of dim should lie in some partially-ordered set. To be "reasonable", the values should be (at least) real numbers. Suppose we have a real-valued dimension concept with the stated properties. Since $\dim \mathbb{R}^0 = 0$, property (3) gives $\dim E \geqq 0$ for all nonempty E.

Let $A \subset \mathbb{R}$ be any closed set with an accumulation point, say, a. The zero function from $\mathbb{R} \to \mathbb{R}^0$ is, by any definition, critical at a on A, so property (4) gives $\dim A > 0$.

We see at once that an integer-valued dimension is impossible. The sets D and E of our construction above would have $0 < \dim E < \dim D \leqq 1$.

An elaboration of this idea rules out even a real-valued dimension. Let α be the infimum of the dimensions of all nondiscrete compact subsets of $\mathbb{R}$. Then $\alpha \geqq 0$. If we can produce a nondiscrete compact set E with dimension α, then we get a contradiction. For then $0<\alpha<1$, and we can define, as in the construction of ψ, a C^∞ map of E onto a nondiscrete F, with ψ critical on E; this makes $0 < \dim F < \dim E = \alpha$, contradicting the minimality of α.

If no E with $\dim E = \alpha$ exists, then there are sets E_n with $\dim E_n \downarrow \alpha$. We may assume that the E_n are perfect and totally-disconnected. For each n, we can construct a C^∞ map $\phi_n : \mathbb{R} \to \mathbb{R}$, flat on E_n, mapping E_n onto a dyadic Cantor set E_n: All we have to do is organise E_n as an intersection of suitable unions of intervals, and pick ϕ_n to have equal integrals over corresponding complementary intervals. Then $\dim B_n < \dim E_n$, so $\dim B_n \downarrow \alpha$.

Now each B_n has a defining sequence $\{d_j^{(n)}\}_{j=0}^{\infty}$. Taking the sequence

$$d_j^0 = j^{-j} \cdot \min \{d_j^{(1)}, \ldots, d_j^{(j)}\}$$

we get a Cantor set B_0 for which each natural map $B_n \to B_0$ has a C^∞ extension, critical on B_n. Thus dim $B_0 <$ dim B_n, so dim $B_0 = \alpha$, and we are done.

If the concept of "reasonable" is relaxed a bit, there is a possibility that there might be a dimension of the kind specified, with values in, say $\mathbb{Z}^+ \times \aleph_1$. A prerequisite for the existence of any such concept is the nonexistence of a closed set A in some C^∞ manifold Y and a C^∞ map $\phi: Y \to Y$ with $\phi | A$ critical on A and $\phi(A) = A$. Here "$\phi | A$ critical at a point $a \in A$ refers only to the action of $D\phi$ on a tangent space to A at a.

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Reference

1. Gromov, M.: Partial differential relations. Berlin-Heidelberg-New York-Tokyo: Springer 1986

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