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Simplified Proof of Vitushkin's Squares Theorem

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Abstract. The paper exploits a new kind of Lip 1 partition of unity on $\mathbb{C}$ and a trick based on Fubini's theorem, to shorten significantly the proof of the fundamental theorem on uniform rational approximation.

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1. The theorem in question [4,p.171, Theorem 2] is the main theorem on qualitative uniform rational approximation:

Theorem. Let X be a compact subset of $\mathbb{C}$ and let $f : \mathbb{C} \to \mathbb{C}$ be uniformly continuous, with modulus of continuity $\omega_f(\delta)$. Then the following conditions are equivalent:

(1) There is a sequence f_n of rational functions with poles off X, such that $f_n \to f$ uniformly on X;

(2) There is a constant $\kappa > 0$ such that

$$\left| \int_{\mathrm{bdy}S} f(z)dz \right| \leq \kappa \, \omega_f(\delta) \, \gamma(S \sim X),$$

whenever S is an open square of side δ. Here γ denotes analytic capacity.

(3) There is a function $\eta(\delta) \downarrow 0$ such that

$$\left| \int_{\mathrm{bdy}S} f(z)dz \right| \leq \eta(\delta) \, \gamma(S \sim X),$$

whenever S is a square of side δ.

The implication (1) $\Rightarrow$ (2) is due to Melnikov.

The deepest part of the theorem is the implication (3) $\Rightarrow$ (1). To prove this part, Vitushkin shows [4,pp.177-180] that condition (3) implies the following:

(4) There exists $\eta(\delta) \downarrow o$ such that

$$\left|\iint f(z) \frac{\partial\phi}{\partial\bar{z}}\, dxdy\right| \leq \eta(\delta) \cdot \delta \cdot \|\nabla\phi\|_\infty \cdot \gamma(E_\phi)$$

whenever $\phi \in \mathcal{D}$ has support in a disc D of diameter δ and E_ϕ denotes the set

$$\left\{z \in \mathbb{C} \sim X : \text{dist}[z, \text{spt}\, \phi] < 2\delta\right\}.$$

He then shows that (4) $\Rightarrow$ (1).

The point of the present note is that there is a simple argument showing that (3) implies a variation (5) (see below) of (4), and that a slight variation of the old argument shows that (5) $\Rightarrow$ (1). Thus the rather hairy argument of [4,pp.177-180] may be dispensed with. Here is the variation of (4):

(5) There exists $\eta(\delta) \downarrow o$ and a constant $\kappa > o$ such that

$$\left|\iint f(z) \frac{\partial\phi}{\partial\bar{z}}\, dxdy\right| \leq \eta(\delta) \cdot \delta \cdot \|\nabla\phi\|_\infty\, \gamma(\kappa S \sim X)$$

whenever $\phi \in$ Lip 1 has support in a square S of side δ and ϕ is constant on each square bdy tS, concentric with and parallel to S.

Basically, (5) is weaker than (4). It is readily seen that (4) implies (5), with $\kappa = 4$.

2. Proof that (3) $\Rightarrow$ (5) (with $\kappa = 1$)

This is the crucial part of the new trick, and the main ingredient is the distribution formula

$$(6) \qquad \frac{\partial\chi_S}{\partial\bar{z}} = -\frac{dz}{2i}\Big|_{\text{bdy } S}$$

valid for any square S (or any set S with rectifiable boundary). This formula is merely a reformulation of Pompeiu's formula.

Let $\phi \in \text{Lip } 1$ have support in a square S of side δ, and be constant on each square bdy tS. Then there is a function $\psi \in L^\infty[o,\delta]$, with $||\psi||_\infty \le ||\nabla\phi||_\infty$, such that

$$\phi(z) = \phi(a) + \delta\int_0^1 \chi_{tS}(z)\,\psi(t)dt,$$

where a is the centre of S. With a little manipulation, formula (6) yields

$$(7) \quad \int f\frac{\partial\phi}{\partial\bar{z}}\,dxdy = \delta\int_0^1\left\{\frac{1}{2i}\int_{\text{bdy } t\,S} f(z)dz\right\}\psi(t)dt,$$

valid for all continuous f on $\mathbb{C}$.

Formula (7) renders it completely obvious that (3) ⇒ (5).

3. Proof that (5) ⇒ (1).

If you look at the argument for (4) ⇒ (1) in Vitushkin's paper [4, Ch.IV, §2, Lemma1], or in Gamelin's book [2,pp.215-217], or at the simplified argument of Davie [1,pp.412-413], you will see that to prove (5) ⇒ (1), it suffices to find a constant $c > 0$ and, for each $\delta > 0$, a partition of unity $\left\{\phi_n^\delta\right\}_{n=1}^\infty$ on C such that

$$(8) \quad \left|\begin{array}{l} 0 \le \phi_n \le 1, \quad \phi_n \in \text{Lip } 1, \quad ||\nabla\phi_n||_\infty \le \frac{c}{\delta} \\ \phi_n \text{ is supported on a square } D_n \text{ of side } \delta \\ \text{no point belongs to more than 9 } D_n\text{'s} \\ \phi_n \text{ is constant on each square bdy } t\,D_n \end{array}\right.$$

and such that, given a number $\beta_n \in (o,\delta)$, we may write $\phi_n^\delta = \sum_k \psi_k^{(n)}$, where the $\psi_k^{(n)}$ satisfy:

$$(9)\quad \left|\begin{array}{l} 0 \le \psi_k^{(n)} \le 1 \quad , \quad \psi_k^{(n)} \in \text{Lip } 1 \quad , \quad \|\nabla\psi_k^{(n)}\|_\infty \le \frac{c}{\beta_n} \\ \\ \psi_k^{(n)} \text{ is supported in a square } E_k^{(n)} \text{ of side } \beta_n \\ \\ \text{no point belongs to more than 9 } E_k^{(n)}\text{'s for fixed } n \\ \\ \psi_k^{(n)} \text{ is constant on each square bdy } t E_k^{(n)}. \end{array}\right.$$

The "variable-size" subpartition $\left\{\psi_k^{(n)}\right\}$ is used for "matching the second coefficient", the heart of all the arguments.

There is, of course, absolutely no problem about constructing such functions $\phi_n^\delta, \psi_k^{(n)}$, except for the restriction that the level sets must be concentric parallel squares. That it can be done, with this restriction, is a recent discovery [3]. Here is an explicit construction: Define

$$\rho(x,y) = \max\left\{0, 1 - \max\left\{|x|, |y|\right\}\right\}$$

$$\sigma(x,y) = \max\left\{0, 1 - \max\left\{|x-y|, |x+y|\right\}\right\}$$

$$\lambda_{m,n}(x,y) = \begin{cases} \rho(x-m, y-n) , & \text{if } m+n \text{ is even} \\ \sigma(x-m, y-n) , & \text{if } m+n \text{ is odd} \end{cases}$$

$$\lambda_{m,n}^\delta(z) = \lambda_{m,n}(z/\delta)$$

$$\left\{\phi_n^\delta\right\}_1^\infty = \left\{\lambda_{m,n}^\delta\right\}_{1\ 1}^{\infty\ \infty} .$$

The $\left\{\phi_n^\delta\right\}$ so defined satisfies conditions (8), with $c = \sqrt{2}$. We now describe how to make the decomposition

$$\phi_n^\delta = \sum_k \psi_k^{(n)}$$

in case $\phi_n^\delta = \lambda_{m,n}^\delta$ with $m+n$ even. The other case is similar. It suffices to consider $\beta = \delta/q$, with q integral. We have, for such β,

$$\lambda_{m,n}^\delta(z) = \sum_{r,s} \phi_n^\delta(\beta r, \beta s)\lambda_{r,s}^\beta(z),$$

and the system $\left\{\psi_k^{(n)}\right\}_k = \left\{\phi_n^\delta(\beta r,\beta s)\lambda_{r,s}^\beta\right\}_{r,s}$ has the properties (9). These facts are readily checked.

The system ϕ_n^δ , $\psi_k^{(n)}$ may be understood geometrically in terms of a subtessallation, by pyramids and tetrahedra of various sizes, of a tessellation by congruent pyramids and congruent tetrahedra of the layer $\{o \leq z \leq 1\}$ in $\mathbb{R}^3$. More detail may be found in [3].

It is worth remarking that Vitushkin's proof requires only that condition (3) of the theorem holds for squares with sides parallel to the coordinate axes, while the new proof requires it also for squares making a 45° angle with the axes.

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