Capacities, analytic and other.

Anthony G. O'Farrell

(1.1) Let E be a compact subset of $\mathbb{C}$. If f is analytic on $S^2 \sim E$, then it has the Laurent expansion

$$f = a_o + \frac{a_1}{z} + \frac{a_2}{z^2} + \ldots$$

near ∞. The (Ahlfors) analytic capacity of E is the non negative number

$$\gamma(E) = \sup |a_1(f)|$$

where f runs over all functions, analytic on $S^2 \sim E$, and bounded by 1 in modulus. A compact set E has $\gamma(E) = 0$ if and only if E is removable for all bounded analytic functions, i.e. if and only if given U open and $f: U \sim E \to \mathbb{C}$, analytic and bounded, there exists an analytic continuation of f to U. For open sets U, $\gamma(U)$ is defined as

$$\sup \{\gamma(E): E \subset U, E \text{ compact}\}.$$

For arbitrary sets $A \subset \mathbb{C}$, the outer analytic capacity $\gamma^*(A)$ is defined as

$$\inf \{\gamma(U): A \subset U, U \text{ open}\}.$$

(1.2) Analytic capacity plays a key role in the theory of uniform rational approximation (or, what amounts to the same thing, holomorphic approximation) in one variable. Let $\mathcal{O}(E)$ denote the set of functions, holomorphic near E. For X compact in $\mathbb{C}$, let R(X) denote the set of uniform limits on X of elements of $\mathcal{O}(X)$. Vitushkin showed that a necessary and sufficient condition that all functions continuous on X belong to R(X) is that

$$\gamma(U \sim X) = \gamma(U)$$

for all open sets U (or, equivalently, for all open discs U). The capacity γ, in combination with another, the continuous analytic capacity α, provides a similar resolution (also due to Vitushkin) of the problem of which X have

$$R(X) = \{f: f \text{ is continuous on } X \text{ and analytic on int } X\}.$$

See [G] for other uses of γ in connection with R(X).

(1.3) There are two important open questions about γ. The first is to give a reasonable "real-variable" characterisation of the γ-null sets. For instance, Vitushkin has conjectured that $\gamma(E) = 0$ if and only if almost all projections of E on lines have outer length zero. Thanks to some work of Havinson, Calderon and others, we know this is true for σ-rectifiable sets, and for those totally unrectifiable sets known to be γ-null [M]. This problem is particularly irritating becuase the bounded analytic functions are practically the only "reasonable" class of analytic functions for which the null sets lack a real-variable description. For instance, see [C]. The only significant exception are the Smirnov E_p classes, but they do not count, because, when defined, they have the same null sets as γ[H].

The second problem is whether γ is <u>quasi-subadditive</u>, i.e. whether there exists a universal constant $\kappa > 0$ such that

$$\gamma(E_1 \cup E_2) \leq \kappa\{\gamma(E_1) + \gamma(E_2)\}$$

whenever E_1 and E_2 are compact in $\mathbb{C}$. There is a sizeable logjam in uniform holomorphic approximation theory because of this problem. For example, if E is compact, with $\gamma(E) = 0$, and $f:S^2 \to \mathbb{C}$ is continuous, do there exist functions $f_n:S^2 \to \mathbb{C}$, tending uniformly to f on S^2, holomorphic wherever f is and on a neighbourhood of E? If γ is quasi-subadditive, the answer is yes. If γ were subadditive, one could define a special topology (the "analytic-fine topology") on $\mathbb{C}$, finer than the Euclidean topology, that ought to be especially helpful for studying R(X). This topology might provide the real answer to E. Borel's dream of the perfect notion of analytic function.

(1.4) The most penetrating work on the subadditivity problem is in [D]. Davie showed that quasisubadditivity would follow from the statement:

$$\gamma^*(E \cup F) \leq \gamma(E) + \kappa(E)\gamma(F)$$

wherever E is compact and F is open, where $\kappa(E) > 0$ is independent of F. We know that

$$\gamma(E \cup F) \leq \gamma(E) + \kappa(E)\gamma(F)$$

wherever E and F are compact. It may not seem like much of a gap, but there it is.

In what follows, we shall present another formula for $\gamma(E)$, and use it to cast a little light on the subadditivity problem. It will become clear that subadditivity is just another version of the only real "problem" in analysis, which is how to handle

$$\int_{-1}^{1} \frac{f(t)}{t}\, dt.$$

(2.1) Dolzenko generalised the concept of analytic capacity. Suppose B is a Banach space of functions on $\mathbb{C}$, such that $\mathcal{D} \hookrightarrow B$, $\mathcal{D} \hookrightarrow B^*$, and the inclusions are continuous. Here $\mathcal{D} = \mathcal{D}(\mathbb{C},\mathbb{C})$ denotes the space of test functions. We assume that, if B has a predual B_*, then $\mathcal{D} \hookrightarrow B_*$, continuously. Also, we assume $f \in B \Rightarrow \bar{f} \in B$. The <u>analytic</u> B-<u>capacity</u> of a compact $E \subset \mathbb{C}$ is

$$\gamma_B(E) = \sup |a_1(f)|$$

where f runs over all functions in the unit ball of B that are analytic on $S^2 \sim E$.

Examples are $B = L_p$ (wrt area measure $\mathcal{L}^2$), C (for continuous and bounded), $\mathrm{Lip}\,\alpha$, $\mathrm{lip}\,\alpha$, BMO, VMO, C^k (= bounded continuous derivatives up to order k), some weighted L_p spaces, Sobolev spaces, etc.

(2.2) The number $a_1(f)$ equals

$$\frac{1}{2\pi i} \int_{\Gamma} f(\zeta)\, d\zeta$$

whenever Γ is a rectifiable contour around E, in the usual sense. A more entertaining formula is

$$a_1(f) = -\frac{1}{\pi} \int f(z)\, \frac{\partial\psi}{\partial\bar{z}}\, d\mathcal{L}^2(z) = \frac{1}{\pi} \left\langle \psi, \frac{\partial f}{\partial \bar{z}} \right\rangle$$

where $\psi \in \mathcal{D}$ is any test function with $\psi = 1$ on a neighbourhood of E. This follows from Green's formula. It suggests the natural way to generalise γ_B from the Cauchy-Riemann operator to other differential operators.

Let E be a compact subset of $\mathbb{R}^d$, let B be a Banach space of functions on $\mathbb{C}$, and let $L : \mathcal{E}(\mathbb{R}^d,\mathbb{C}) \to \mathcal{E}(\mathbb{R}^d,\mathbb{C})$ be a linear differential operator with C^∞ coefficients. Choose $\psi \in \mathcal{D}(\mathbb{R}^d,\mathbb{C})$ with $\psi = 1$ on a neighbourhood of E, and define

$$\gamma_B^L(E) = \sup \left| \int f(x)\, L^*\psi(x)\, d\mathcal{L}^d(x) \right|,$$

where f runs over all elements of the unit ball of B which satisfy $Lf = 0$ on $\mathbb{R}^d \sim E$, in the (weak) sense of distributions. The value of the integral does not depend on the choice of ψ, for such f. This concept embraces those capacities used by Hedberg, Polking, Bagby, and others in connection with various approximation problems. The classical Newtonian capacity is $\gamma_{L^\infty}^{\Delta}$, where Δ is the Laplacian.

(2.3) The technique of the *dual extremal problem* is based on the following fact, which may be proved by using the Hahn-Banach theorem.

Duality Lemma. *Let* B_o *be a subspace of a Banach space* B, *and let* $A \in B^*$.

(1) Then

$$\sup \{|Af| : f \in B_o, \|B\|_B \le 1\} = \text{dist}\,(A, B_o^{\perp}).$$

(2) *If* B *has a predual* B_*, *if* B_o *is* $B_1^{\perp}$, for some subspace $B_1 \subset B_*$, *and if* $A \in B_*$, *then*

$$\sup \{|Af| : f \in B_o, \|f\|_B \le 1\} = \text{dist}\,(A, B_1).$$

This lemma allows us to turn an extremal problem in one Banach space into a corresponding problem in the dual, or in the pre-dual (if there is a pre-dual). This technique has been put to good use in the past, but still has plenty of energy left. Our present purpose is to apply it to get formulas for the kind of capacities described above, so as to cast some light on the subadditivity problem.

(2.4) Applying part (1) of the Duality Lemma gives the formula

$$\gamma_B^L(E) = \inf_S \|L^*\psi - S\|_{B_*}$$

where S runs over all elements of B* such that

$$\left.\begin{array}{l} f \in B \\ Lf = 0 \text{ off } E \end{array}\right\} \Rightarrow Sf = 0.$$

If B has a predual B_* (and $D \hookrightarrow B_*$ is continuous), part (2) gives the nicer formula

$$\gamma_B^L(E) = \inf_\phi \|L^*\psi - L^*\phi\|_{B_*}$$

where ϕ runs over all test functions supported on $\mathbb{R}^d \sim E$. Recalling that ψ is any given test function with $\psi = 1$ near E, we conclude that

$$\gamma_B^L(E) = \inf\{\|L^*\phi\|_{B_*} : \phi \in \mathcal{D},\ \phi = 1 \text{ near } E\}.$$

(2.5) Applying this formula to classical analytic capacity, we get

$$\gamma(E) = \frac{1}{\pi} \inf \{\|\frac{\partial\phi}{\partial\bar{z}}\|_{L_1} : \phi \in \mathcal{D},\ \phi = 1 \text{ near } E\}.$$

(2.6) Applying it to the analytic capacity associated to $B = L_p$ (the "analytic p-capacity" of Sinanjan), we get

$$\gamma_{L_p}(E) = \frac{1}{\pi} \inf \{ \|\frac{\partial\phi}{\partial\bar{z}}\|_{L_q} : \phi \in \mathcal{D},\ \phi = 1 \text{ near } E\}$$

for $1 < p < \infty$, where q is the conjugate index to p. This B has the property that B is mapped continuously to itself by the Beurling transform:

$$(Tf)(z) = \frac{1}{\pi} \int \frac{f(\zeta)}{(\zeta - z)^2} d\mathcal{L}^2(\zeta),$$

where the integral is interpreted as a limit in B norm of principal value integrals of smooth approximations to f. The theory of the continuity properties of this and similar integral operators is known as the Calderon-Zygmund theory. The operator T has the property that

$$T \frac{\partial\phi}{\partial\bar{z}} = \frac{\partial\phi}{\partial z}$$

for all $\phi \in \mathcal{D}$, so that if T maps $B \to B$ continuously, we deduce that γ_B is comparable to the real-variable capacity

$$\inf \{ \|\frac{\partial\phi}{\partial x}\|_{B_*} + \|\frac{\partial\phi}{\partial y}\|_{B_*} : \phi \in \mathcal{D},\ \phi = 1 \text{ near } E \}.$$

Apart from L_p $(1 < p < \infty)$, the spaces Lipα and BMO are Beurling-invariant dual spaces, so this argument also applies to their analytic capacities. In all three cases, this real-variable formula gives a proof of quasi-subadditivity. For instance, for BMO we get

$$\gamma_{BMO}(E) \sim \inf \{ \| \,|\nabla\phi|\, \|_{L_1} : \phi \in \mathcal{D},\ \phi = 1 \text{ near } E \},$$

where ~ means "is within constant multiplicative bounds of". It makes no difference to restrict to real-valued ϕ, and we get

$$\gamma_{BMO}(E) \sim \inf \{ \|\phi\|_{W^{1,1}} : \phi \in \mathcal{D},\ \phi = 1 \text{ near } E \}$$

$$= \inf \{ \|h\|_{W^{1,1}} : h \in W^{1,1},\ h = 1 \text{ near } E \}$$

$$= \inf \{ \| h \|_{W^{1,1}} : h \in W^{1,1},\ h \geq 1 \text{ near } E \};$$

which is obviously subadditive. Here $W^{1,1}$ denotes the Sobolev space of L_1 functions with L_1 distributional derivatives. See [V].

(2.7) This method extends to other hypoelliptic operators. Suppose L* has an inverse $P : \mathcal{D} \to \mathcal{E}$ such that $PL\phi = \phi$ whenever $\phi \in \mathcal{D}$. For instance, the Cauchy transform does this for $\frac{\partial}{\partial\bar{z}}$, and, more generally, convolution with a fundamental solution does it for elliptic constant-coefficient L.

Suppose L has order m. Denoting the partial derivative associated to the multiindex j by D_j, we may ask about the continuity properties with respect to B of the operator $D_j P$, for $|j| \leq m$. If all these map B_* continuously into B_*, then $\gamma_B^L(E)$ is comparable to the real-variable capacity

$$\inf \{ \sum_{|j|\leq m} \|D_j \phi\|_{B_*} : \phi \in \mathcal{D}, \phi = 1 \text{ near } E\}.$$

This works for constant-coefficient elliptic operators, with $B = L_p$ $(1 < p < \infty)$, Lipα, BMO, Lip$(k+\alpha)$, some Sobolev spaces, etc. The associated γ_B^L are then subadditive.

(2.9) If L has <u>real-valued</u> coefficients, then γ_B^L is a real-variable capacity even if B is not Beurling invariant. For instance,

$$\begin{aligned}\gamma_{L_\infty}^{\Delta}(E) &= \inf \{\| \Delta\phi\|_{L_1} : \phi \in \mathcal{D}, \phi = 1 \text{ near } E\} \\ &= \inf \{\| \Delta\phi\|_{L_1} : \phi \in \mathcal{D}, \phi = 1 \text{ near } E, \phi \text{ real}\} \\ &= \inf \{\| \Delta\phi\|_{L_1} : \phi \in \mathcal{D}, \phi \geq 1 \text{ near } E, \phi \text{ real}\}.\end{aligned}$$

This is pretty clearly subadditive.

(2.10) The upshot is that among the usual crop of elliptic operators L and dual spaces B, the case $L = \bar{\partial}$ and $B = L_\infty$ is practically the only one we cannot handle with ease. And we cannot handle it at all.

References

[C] L. Carleson, Selected Problems on Exceptional Sets, Van Nostrand 1967.

[D] A. M. Davie, Analytic capacity and approximation problems, Transactions AMS 171 (1972) 409 - 444.

[G] T. W. Gamelin, Uniform Algebras, Prentice-Hall, 1969.

[H] S. Ya. Havinson, Analytic capacity of sets ..., AMS Translations (2) 43 (1964) 215 - 266.

[M] D. Marshall, Removable sets for bounded analytic functions, in Contemporary Problems in Complex and Linear Analysis, ed. S. N. Nikolshii, Nauka 1980.

[V] J. Verdera, Rational approximation in BMO norms and one-dimensional Hausdorff content, to appear.

Maynooth College
Co. Kildare
Ireland.