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THE LEGENDRE EXPANSION OF A SMOOTH FUNCTION

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Dedicated to Academician Ljubomir Iliev on the occasion of his seventieth birthday

1. Introduction. Each function $f \in L_2(-1,1)$ has a Legendre expansion $f=\Sigma_{n=0}^{\infty} c_n(f)P_n$, converging in L_2 norm. This paper is about the convergence of this series in bigger norms, for smooth functions f. In particular, we show that if $f \in C^\infty$, then all term-by-term derivatives of the series converge uniformly on $[-1, 1]$, and the rate of convergence of each derived series is faster than n^{-k}, for all k. The main result is a polynomial approximation theorem, stated in section 3. ∧ε

We came to investigate the Legendre series because we are interested in the simultaneous approximation of a function and all its derivatives by polynomials and their derivatives. A p o l y n o m i a l a p p r o x i m a t i o n s c h e m e on the interval $I=[a, b]$ is a sequence A_n of maps from the space $C^\infty(I)$ to the space of polynomials of degree at most n, such that $A_n f(x) \to f(x)$ as $n \uparrow \infty$ for each $x \in I$ and each $f \in C^\infty$. We seek schemes such that $(A_n f)^{(k)} \to f^{(k)}$ for each k. There are many such schemes, as one may easily deduce from the Weierstrass approximation theorem. We seek schemes which are readily computable and give good results. This raises the question of how to evaluate a scheme. Since the best uniform n-th degree polynomial approximation to a C^k function approximates it to order n^{-k}, we expect that a good scheme should satisfy

$$n^r \| f^{(k)} - (A_n f)^{(k)} \|_\infty \to 0 \tag{1.1}$$

as $n \uparrow \infty$, for all r and k. This straightaway rules out the Bernstein scheme, the Jackson schemes, and all positive linear schemes, because of saturation properties. We could ask for more. Given positive numbers M_k such that

$$M_k^{-1} \| f^{(k)} \|_\infty \to 0 \tag{1.2}$$

as $k \uparrow \infty$, we could ask whether there exist $A_n f$ such that

$$\sup_k M_k^{-1} \| f^{(k)} - (A_n f)^{(k)} \|_\infty \to 0 \tag{1.3}$$

as $n \uparrow \infty$. We call this the B question. An easier question is whether, given that

$$\lambda^k \cdot\cdot M_k^{-1} \| f^{(k)} \|_\infty \to 0 \tag{1.4}$$

one dot

for all $\lambda>0$, there exists a scheme such that

$$\sup_k \lambda^k M_k^{-1} \| f^{(k)}-(A_n f)^{(k)} \|_\infty \to 0 \tag{1.5}$$

for all $\lambda>0$. We call this the F question.

It is possible to prove that the best uniform scheme has property (1.1), as does each best C^k scheme. These schemes are nonlinear, and difficult to compute. The simplest linear unsaturated scheme is the Legendre series expansion. In section 2, we derive some identities and estimates for this expansion, which show that it has property (1.1). The Tchebyshev series also has this property, as anyone acquainted with the equiconvergence properties of Jacobi series might expect.

In section 3 we use the Legendre series to prove that the F question has a positive answer (although the Legendre sheme itself may not have property (1.5)). Hitherto, for certain M_k (the "nonquasianalytic and logarithmically convex"), this result could be deduced from the work of C.-C. Chou [2], whereas for other M_k (the "quasianalytic") we had only a rather complicated (unpublished) proof, based on T. Carleman's work [1]. The harder B question has been answered only for some M_k, and will be discussed elsewhere.

The corresponding results for Fourier trigonometric series of periodic functions are trivial. The reason is that the characters exp $(in\pi x)$ are eigenfunctions of differentiation, so that the Fourier series of f' is the term-by-term derived series of f, for smooth periodic f. The action of differentiation on the Legendre coefficients is a good deal more complicated, in fact

$$c_n(f')=(2n+1)(c_{n+1}(f)+c_{n+3}(f)+\dots).$$

The corresponding problems on the whole real line, involving approximation by entire functions, are also trivial. The problem with a bounded interval is created by the presence of the endpoints.

2. Identities and Estimates for Legendre series. The Legendre polynomials are given by Rodriguez' formula:

$$P_n(x)=\frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n, \quad n\geqq 0.$$

The functions $(n+1/2)^{1/2}P_n$ form a complete orthonormal basis for $L_2(-1,1)$, and hence the representation $f(x)=\Sigma_{n=0}^{\infty}c_n(f)P_n(x)$ sets up an isometric isomorphism between L_2 and the weighted l_2 space with weights $(n+1/2)^{-1}$. We treat functions f and sequences $c=(c_0, c_1, c_2, \dots)$ as interchangeable objects.

Consider the following (possibly unbounded) operators:

$$Df=f', \qquad Sc=(0, c_0, c_1, \dots),$$
$$Nc=(0, c_1, 2c_2, 3c_3, \dots), \qquad Tc=(c_1, c_2, c_3, \dots).$$

The adjoint of S is given by

$$S^*\{c_n\}=\{\frac{2n+1}{2n+3}c_{n+1}\},$$

so that $(2N+3I)S^*=(2N+I)T$, where I denotes the identity.

The coefficient $c_n(f)$ is given by

$$c_n(f)=(n+1/2)\int_{-1}^{1} fP_n dx. \tag{2.1}$$

In view of the relations [6, p. 302 and p. 308] $P_n(1)=1$, $P_n(-1)=(-1)^n$, $P'_{n+1}=xP'_n+(n+1)P_n$, we have

$$c_{n+1}(f')/(n+3/2)=\int_{-1}^{1} f'P_{n+1}dx=f(1)+(-1)^{n+2}f(-1) \tag{2.2}$$

$$-\int_{-1}^{1}(xP'_n+(n+1)P_n)f\,dx=(c_n(Xf')-nc_n(f))/(n+1/2),$$

$$N=(X-S^*)D. \tag{2.3}$$

The recursion relation [6, p. 308] $(n+1)P_{n+1}-(2n+1)xP_n+nP_{n-1}=0$ yields the operator equation $X(2N+I)=S(N+I)+TN$, so that the action of X on sequences is given by $Xc=\{nc_{n-1}/(2n-1)+(n+1)c_{n+1}/(2n+3)\}$. Consequently, $(X-S^*)c=\{nc_{n-1}/2n-1-nc_{n+1}/2n+3\}$. By (2.3) we have

$$c_n(f)=c_{n-1}(f')/(2n-1)-c_{n+1}(f')/(2n+3),\quad n\geqq 1. \tag{2.4}$$

Telescoping, we get

$$\frac{c_{n+1}(f')}{2n+3}=-c_n(f)-c_{n-2}(f')-\dots\begin{cases}-c_1(f)+c_0(f'), & n \text{ odd},\\ -c_2(f)+\frac{1}{3}c_1(f'), & n \text{ even}.\end{cases}$$

But

$$c_0(f')=\frac{1}{2}\int_{-1}^{1}f'dx=\frac{1}{2}(f(1)-(f(-1))=\sum_{n=0}^{\infty}\frac{1}{2}c_n\cdot(P_n(1)$$

$$-P_n(-1))=\sum_{m=0}^{\infty}c_{2m+1},$$

and similarly $c_1(f)=3\Sigma_{m=1}^{\infty}c_{2m}$, so that in each case $c_n(f')=(2n+1)(c_{n+1}(f+c_{n+3}(f)+\dots)$, as stated in the introduction.

By (2.1) and the Cauchy inequality, $|c_n(f)|\leqq(n+1/2)^{1/2}\|f\|_2$. By (2.4), iterated k times,

$$|c_n(f)|\leqq\frac{2^k\max_{-k\leqq j\leqq k}|c_{n+j}(f^{(k)})|}{(2n-1)(2n-3)\dots(2n-2k+1)}\leqq\frac{2^k(n+k+1/2)^{1/2}\|f^{(k)}\|_2}{(2n-1)\dots(2n-2k+1)}. \tag{2.5}$$

By [5, p. 63], the k-th derivative of P_n is $2^{-k}(n+1)(n+3)\dots(n+2k-1)P^{(k,k)}_{n-k}$, where $P^{(\alpha,\beta)}_m$ denotes a Jacobi polynomial. By [5, p. 168],

$$\|P^{(k,k)}_{n-k}\|_\infty=\binom{n}{n-k}.$$

Thus

$$\|P^{(k)}_n\|_\infty=2^{-k}(n+1)\dots(n+2k-1)\binom{n}{n-k}. \tag{2.6}$$

We now see that for fixed k, $c_n(f)$ is of order $n^{-k+1/2}\|f^{(k)}\|_2$, and $\|P^{(k)}_n\|_\infty$ is of order n^{2k}, hence the series $\Sigma_{n=0}^{\infty}c_n(f)P^{(k)}_n$ converges uniformly on $[-1, 1]$ whenever $f\in C^{2k+2}(-1,1)$.

The argument for Tchebyshev series goes in the same manner, using the fact [5, p. 63] that $T^{(k)}_n=2^{-k}n(n+2)\dots(n+2k-2)P^{(k-1/2,\,k-1/2)}_{n-k}$, where T_n is the n-th Tchebyshev polynomial, so that $\|T^{(k)}_n\|_\infty$ is of order $n^{2k-1/2}$.

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f'

These results may also be deduced by using Markov's inequality [3, p. 137] to estimate $\| P_n^{(k)} \|_\infty$ and $\| T_n^{(k)} \|_\infty$. The same method shows that the best uniform scheme satisfies (1.1).

3. Polynomials Dense in Some Fréchet Spaces of Smooth Functions. In this section we show that, given $f \in C^\infty(I)$ and $M=\{M_k\}$ satisfying (1.4) for all $\lambda>0$, there exists a sequence $A_n f$ of polynomials satisfying (1.5) for all $\lambda>0$.

Let $I=[a, b]$ and let M be a sequence of positive numbers. Let $F(M, I)$ denote the space of all functions $f \in C^\infty(I)$ such that (1.4) holds for all $\lambda>0$. Then $F(M, I)$ is a Fréchet space, with the topology induced by the seminorms

$$\sup_k \lambda^k M_k^{-1} \| f^{(k)} \|_\infty,$$

where λ runs over all positive numbers. We may formulate the desired result as follows.

Theorem. *The polynomials are dense in $F(M, I)$, for each sequence M and each compact interval I.*

Lemma 1. *Let $N_k = \min\{M_{k-1}, M_k\}$. Then $F(N, I)$ is a dense subset of $F(M, I)$.*

P r o o f. Without loss of generality, we may take $I=[-1, 1]$. Since $N_k \leqq M_k$, it follows that $F(N, I) \subset F(M, I)$.

Let $f \in F(M, I)$. For $0<\eta<1$, define

$$g_\eta(x) = 2\eta^{-1} \int_{x-\eta-\eta x}^{x+\eta-\eta x} f(t)\,dt.$$

Then it is easy to check that $g_\eta \in F(N, I)$ and $g_\eta \to f$ in $F(M, I)$ topology as $\eta \downarrow 0$. This proves the lemma.

Let αI denote the interval with the same midpoint as I and α times the length of I.

Lemma 2. *The space $\bigcap_{\alpha>1} F(M, \alpha I)$ is dense in $F(M, I)$.*

P r o o f. It suffices to consider $I=[-1, 1]$. Let $f \in F(M, I)$ be given.

Define $g_\alpha(x)=f(\alpha^{-1}x)$. Then one easily verifies that $g_\alpha \in F(M, \alpha I)$ and $g_\alpha \to f$ in $F(M, I)$ topology as $\alpha \downarrow 1$.

P r o o f o f t h e o r e m. By Lemmas 1 and 2, it suffices to show that if $N_k = \min\{M_{k-1}, M_k\}$, and $\alpha>1$ and $f \in F(N, \alpha I)$, then f is a limit of polynomials in $F(M, I)$ topology. Without loss of generality, we may take $\alpha I=[-1, 1]$, so that $I=[-1+\varkappa, 1-\varkappa]$ for some $\varkappa$, $0<\varkappa<1$. It suffices to show that the Legendre series of f converges to f in the topology of $F(M, I)$.

By [5, p. 208], $P_n^{(k)}(\cos\theta)$ is equal to

$$2\alpha_n \sum_{\nu=0}^{k-1} \alpha_\nu \frac{(1-k)(2-k)\ldots(\nu-k)\cos\{(n-\nu+k)\theta-(\nu+k)\pi/2\}}{(n+k-1)(n+k-2)\ldots(n+k-\nu)(2\sin\theta)^{\nu+k}},$$

where $\alpha_n=\binom{n+k-1}{n}$. Thus

$$|P_n^{(k)}(\cos\theta)| \leqq \frac{2^{1-k}}{|\sin\theta|^{2k-1}} \sum_{\nu=0}^{k-1} \frac{\alpha_n \alpha_\nu (k-1)\ldots(k-\nu)}{(n+k-1)\ldots(n+k-\nu)}$$

$$= \frac{2^{1-k}}{|\sin\theta|^{2k-1}} \sum_{\nu=0}^{k-1} \binom{n+k-\nu-1}{n}\binom{\nu+k-1}{\nu}.$$

For $n \geqq k/2$ we have (using Stirling's inequality [6, p. 251])

$$\binom{n+k}{n} = (1+k/n)\ldots(1+k/1) = e^k(1+1/2+\ldots+1/n)\prod_{j=1}^{n}(1+k/j)e^{-k/j}$$

$$\leq e^{k(1+\log n)}\Gamma(k)^{-1}e^{-\gamma k}k^{-1}\prod_{j=n+1}^{\infty}(1+k/j)^{-1}e^{k/j}$$

$$\leq e^k n^k k^{-k-1/2}e^k(2\pi)^{-1/2}e^{-\gamma k}\prod_{j=n+1}^{\infty}\{1+\frac{k^2}{2j^2}+\frac{k^4}{4!j^4}+\dots\}$$

$$\leq e^{(2-\gamma)k}n^k k^{-k-1/2}\exp\{\prod_{j=n+1}^{\infty}k^2/j^2\}\leq e^{(2-\gamma)k}n^k k^{-k-1/2}e^{k^2/n}.$$

For $n\leq k$ we use

$$\binom{n+k}{n}\leq(2k)^n.$$

For $0\leq\nu<(k-1)/2$, $1\leq k\leq n$, we have

$$(3.2)\quad \binom{n+k-\nu-1}{n}\binom{\nu+k-1}{\nu}\leq e^{(2-\gamma)(k-\nu-1)}n^{k-\nu-1}(k-\nu-1)^{-k+\nu+1/2}e^{(k-\nu-1)^2/n}2^{\nu}(k-1)^{\nu}$$

$$\leq e^{(3-\gamma)k}n^{k-1}\left(\frac{k-1}{2}\right)^{-k+\nu+1/2}2^{\gamma}(k-1)^{\nu}\leq(2e^{3-\gamma})^k n^{k-1}.$$

For $(k-1)/2\leq\nu\leq k-1$, $1\leq k\leq n$, we have

$$(3.3)\quad \binom{n+k-\nu-1}{n}\binom{\nu+k-1}{\nu}\leq e^{(2-\gamma)(k-\nu-1)}n^{k-\nu-1}(k-\nu-1)^{-k+\nu+1/2}e^{(k-\nu-1)^2/n}$$

$$\times e^{(2-\gamma)(k-1)}\nu^{k-1}(k-1)^{-k-1/2}e^{(k-1)^2/\nu}\leq e^{(7-2\gamma)k}n^{k-1}.$$

By (3.1), (3.2), and (3.3),

$$(3.4)\ |P_n^{(k)}(\cos\theta)|\leq 2^{1-k}|\sin\theta|^{-2k+1}\{(2e^{3-\gamma})^k k/2+e^{(7-2\gamma)k}k/2\}n^{k-1}\leq e^{7k}n^{k-1}|\sin\theta|^{-2k+1}$$

For $\cos\theta\in[-1+\varkappa,\ 1-\varkappa]$, we have $|\sin\theta|\geq(2\varkappa-\varkappa^2)^{1/2}$. Combining this with (2.5) and (3.4), we obtain

$$|\sum_{n=0}^{\infty}c_n(f)P_n^{(k)}|=|\sum_{n=k}^{\infty}c_n(f)P_n^{(k)}|\leq\sum_{n=k}^{\infty}\frac{2^{k+1}\ (n+k+3/2)^{1/2}\,\|f^{(k+1)}\|_2}{(2n-1)\dots(2n-2k-1)}\cdot\frac{e^{7k}n^{k-1}}{(2\varkappa-\varkappa^2)^k}$$

$$\leq\{\sum_{n=k}^{2k+1}\frac{100(3k+3)^{1/2}k^{k-1}}{k!}+\sum_{n=2k+2}^{\infty}\frac{2^{k+1/2}}{n^{3/2}}\}\frac{e^{7k}\|f^{(k+1)}\|_2}{(2\varkappa-\varkappa^2)^k}\leq\frac{300\cdot2^k e^{8k}}{(2\varkappa-\varkappa^2)^k}\|f^{(k+1)}\|_2,$$

on $[-1+\varkappa,\ 1-\varkappa]$. Thus on $[-1+\varkappa,\ 1-\varkappa]$, we have, for each $\lambda>0$,

$$\lambda^k M_k^{-1}\|\sum_{n=0}^{\infty}c_n(f)P_n^{(k)}\|_\infty\leq 500\left(\frac{2e^8\lambda}{2\varkappa-\varkappa^2}\right)^k N_{k+1}^{-1}\|f^{(k+1)}\|_\infty\to0,$$

as $k\uparrow\infty$. This completes the proof.

As regards C^k convergence, for fixed k, we would expect that the results implicit here could be sharpened considerably, in view of P. Suetin's results [4] for uniform convergence.

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