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HOLOMORPHIC APPROXIMATION IN LIPSCHITZ NORMS*

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1. INTRODUCTION

For basic material, see [6,7,11,18,23].

Let $X \subset \mathbb{C}^n$ be compact, and let $\mathcal{O}(X)$ denote the space of complex-valued functions, holomorphic on a neighborhood (depending on the function) of X. In order that $\mathcal{O}(X)$ be dense in C(X), the uniform algebra of all continuous functions on X, it is necessary that X be holomorphically-convex (i.e. that X coincide with the set of nonzero algebra homomorphisms of $\mathcal{O}(X) \to \mathbb{C}$), and have no interior. It is also necessary that X contain no nontrivial (i.e. positive-dimensional) analytic subvariety of $\mathbb{C}^n$, and, for this reason, efforts to derive sufficient conditions have centered around the study of totally-real sets. A set $A \subset \mathbb{C}^n$ is totally-real if each point has a neighborhood N in $\mathbb{C}^n$ such that $A \cap N$ is a subset of a C^1 submanifold of $\mathbb{C}^n$ having no complex tangents. For locally-compact A, this is the same as saying that each point has a neighborhood N on which there is defined a C^2 nonnegative strictly plurisubharmonic function, vanishing precisely on $A \cap N$ [9].

Naturally, it is far from necessary that X be totally-real, in order that $\mathcal{O}(X)$ be dense in C(X). Having a few complex tangents is a long way from containing a nontrivial analytic variety. In one variable, where Vitushkin [20, 6] has completely solved the problem, there are examples of sets X whose C^1 tangent space (the space of bounded point derivations on the quotient of the Whitney algebra $C^1(X)$ by its radical) has dimension 2 at each point, whereas $\mathcal{O}(X)$ is dense in C(X). Thus, one is led to conjecture that not only uniform, but "better than uniform" approximation should be possible on totally-real sets, and that one ought also to be able to handle sets having modest "singular subsets", on which they are not totally-real.

*Dedicated to C.E. Rickart on the occasion of his retirement.

Range and Siu [17] proved that if X is a C^k totally-real bordered submanifold, then $\mathcal{O}(X)$ is dense in $C^k(X)$. See also [5,8,10,12,19,23]. If, however, the manifold has even one complex tangent, then C^1 approximation fails, for obvious reasons. This suggests that for X having occasional complex tangents one could profitably look at the $\mathrm{Lip}(\alpha,X)$ norms $(0<\alpha<1)$, which interpolate between the uniform and C^1 norms. For Lipschitzian graphs X in $\mathbb{C}^2$, two of us proved [14] that if the set of points where X has complex tangents has (Hausdorff) area zero, then uniform polynomial approximation implies Lip α polynomial approximation. In [15], we considered polynomially-convex graphs in $\mathbb{C}^{2n}$, totally-real off a closed exceptional set E, and we showed that

$$\mathrm{clos}_{\mathrm{Lip}(\alpha,X)}\mathcal{O}(X) = \mathrm{lip}(\alpha,X) \cap \mathrm{clos}_{\mathrm{Lip}(\alpha,E)}\mathcal{O}(X). \tag{1}$$

Our present purpose is to extend this result to cover general holomorphically-convex sets X.

THEOREM 1. <u>Let the compact set</u> $X \subset \mathbb{C}^n$ <u>be holomorphically-convex</u>. <u>Let</u> $E \subset X$ <u>be closed, and suppose that each point</u> $a \in X \sim E$ <u>has a neighborhood</u> N <u>in</u> $\mathbb{C}^n$ <u>such that</u> $X \cap N$ <u>is a subset of a</u> C^1 <u>submanifold having no complex tangents</u>. <u>Then</u> (1) <u>holds for</u> $0<\alpha<1$.

The space $\mathrm{Lip}(\alpha,X)$ has the norm

$$\sup_X |f| + \sup\left\{\frac{|f(x)-f(y)|}{|x-y|^\alpha} : x,y \in X,\ x \neq y\right\},$$

and $\mathrm{lip}(\alpha,X)$ is the closed subspace in which

$$\sup_{0<|x-y|<\delta} \frac{|f(x)-f(y)|}{|x-y|^\alpha} \to 0$$

as $\delta \downarrow 0$.

In case X is a bordered submanifold and E is empty, the hypothesis that X be holomorphically-convex follows from the other hypothesis [8]. This case of the theorem follows from the Range-Siu theorem, since C^1 convergence implies Lip α convergence on nice sets.

The compact sets which are intersections of (Euclidean) Stein neighborhoods form a proper subclass of the holomorphically-convex compact sets. They are called <u>holomorphic sets</u>. In general, a holomorphically-convex set is an intersection of projections of Stein Riemann domains [2]. A sufficient condition for X to be holomorphic is that it be rationally-convex. Another sufficient condition [8, 10] is the existence of a C^2 strictly plurisubharmonic function ρ on a neighborhood W of bdy X such that

$X \cap W = \{\rho < 0\}$. (Note that the interior of X is not assumed empty in Theorem 1.)

We prove the theorem by using duality, combining the method of Berndtsson [1] with the technique of [15]. We remark in passing that Berndtsson's method also proves the analogue of Theorem 1 for uniform approximation. Weinstock (e.g. [22]) has proved some cases of this theorem. The statement is as follows.

THEOREM 2. Let X be a holomorphically-convex set, let E be a closed subset of X, and let $X \sim E$ be totally-real. Then

$$\text{clos}_{C(X)}\, \mathcal{O}(X) = C(X) \cap \text{clos}_{C(E)}\, \mathcal{O}(X). \tag{2}$$

This result is also implicit in the constructive work of Henkin and Leiterer [10], but the duality proof is simpler. Of course, Theorem 2 is a corollary of Theorem 1.

2. PROOF OF THEOREM 1

Let $T \in \text{Lip}(\alpha,X)^*$ annihilate $\mathcal{O}(X)$. In the same way as in [15] it suffices to show that the distribution $T|C_0^\infty$ is supported on E. Briefly, this reduction depends on three facts: (1) C^∞ functions are dense in $\text{lip}(\alpha,X)$, (2) there is a continuous extension operator from $\text{lip}(\alpha,E)$ to $\text{lip}(\alpha,X)$, and (3) if a $\text{lip}(\alpha,X)$ function vanishes on E, then it is a $\text{Lip}(\alpha,X)$ limit of $\text{lip}(\alpha,X)$ functions which vanish on a neighborhood of E. Thus it suffices to show that each point $a \in X \sim E$ has a neighborhood U in $\mathbb{C}^n$ such that $T\varphi = 0$ wherever $\varphi \in C_0^\infty$ has support in U.

Fix $a \in X \sim E$, and choose a neighborhood N of a such that $X \cap N$ is a subset of a C^1 submanifold M having no complex tangents. Following Berndtsson [1], construct kernels $K(\zeta,z)$ and $\tilde{K}(\zeta,z)$ on $U \times W$, where U is a neighborhood of a and W is a neighborhood of X. Note the following points:

(1) For our present purpose, the set V should be chosen a neighborhood of $C_r \cap M$, not $C_r \cap X$. Next, D should be a neighborhood of X whose holomorphic hull R (which is a Riemann domain) has projection $\pi(R) \subset \mathbb{C}^n$, disjoint from $C_r \sim V$. This is possible, because a holomorphically-convex set X has a sequence of neighborhoods $D_n \downarrow X$ such that the projection $\pi(R_n)$ of the holomorphic hulls R_n of D_n shrink to X. Then, the Cousin problem should be set up on R instead of D, using the covering by the two open sets $\{|\pi| < 2r\}$, $\{|\pi| > r\}$.

(2) Berndtsson refers to Ovrelid [16] for C^1 dependence of the various functions on ζ. However, Ovrelid refers to Hörmander and Bungart. There

are (at least) three published proofs of the desired facts (solubility of Cousin and related problems with smooth dependence on a parameter) - by Bishop [3], Bungart [4], and Weinstock [21]. Of the three, Bishop's method is the most elementary. The others use the powerful Grothendieck tensor product theory.

(3) Berndtsson's function H has Weinstock's "omitted sector property", i.e. for each ζ there exists δ such that $H(\zeta,z)$ takes no value in the sector $\{w \in \mathbb{C} : 0 < |w| < \delta , \ |\mathrm{Im} w| + \delta \,\mathrm{Re} w < 0\}$. (He also needs this fact, to establish the relation

$$\int \bar{\partial}\varphi(\zeta) \wedge \int \tilde{K}(\zeta,z)\,d\mu(z) = 0$$

on page 125.)

Once the kernels are constructed, we proceed as in [15]. Since K is a Cauchy-Fantappié kernel, we get

$$T\varphi = T_z \int_U K(\zeta,z) \wedge \bar{\partial}\varphi(\zeta)$$

whenever $\varphi \in C_0^\infty$ has support in U. Using the DeLeeuw representation of T on lip (α,X) by means of a measure μ on the set of off-diagonal elements of $X \times X$, as in [15], we may write $T\varphi$ as

$$\int_{X\times X} |x-y|^{-\alpha} \int_U \{K(\zeta,x) - K(\zeta,y)\} \wedge \bar{\partial}\varphi(\zeta)\,d\mu(x,y).$$

Now there exists a constant $M_1 < 0$ such that for all $x,y \in X$,

$$\int_U |\{K(\zeta,x) - K(\zeta,y)\} \wedge \omega(\zeta)| \le M_1|x-y|^{\alpha}\,\|\omega\|_\infty$$

for all $(0,1)$ forms ω having bounded measurable coefficients, where $|\nu|$ denotes the total variation of the (n,n) form ν. This is proved just as in [15]. This estimate allows us to apply Fubini's theorem to write $T\varphi$ as

$$\int_U \int_{X\times X} |x-y|^{-\alpha}\{K(\zeta,x)-K(\zeta,y)\}\,d\mu(x,y) \wedge \bar{\partial}\varphi(\zeta).$$

It remains to show that the inner integral vanishes for L^{2n} almost all ζ, and for this it suffices to show that for almost all ζ there exist sequences of functions in $\mathcal{O}(X)$ approximating the coefficients of $K(\zeta,z)$ in an appropriate way, for $z \in X$. This is done by noting that $\tilde{K} = K$ for $z \in X$, and (using the omitted sector property) using $H + m^{-1}$ (m a sufficiently large integer) in the denominator of $\tilde{K}$, instead of H. The details go through in the same manner as in [15].

3. EXAMPLES

(3.1) Let A denote the truncated cone

$$\{(re^{i\theta}, re^{3i\theta}) : \frac{1}{2} \le r \le 2,\ 0 \le \theta \le 2\pi\},$$

and let B denote the torus

$$\{(z,w) \in \mathbb{C}^2 : |z| = |w| = 1\} .$$

If $X = A \cup B$, and $(z_0,w_0) \in \mathbb{C}^2 \sim X$, then at least one of the polynomials $z-z_0$, $w-w_0$, zw_0-z_0w, $z^3w_0-z_0^3w$ is nonvanishing on X. Thus X is rationally convex.

The set $E = A \cap B$ is the curve $\{(e^{i\theta},e^{3i\theta})\}$. The set $X \sim E$ is totally-real, so Theorem 1 shows that the closure of $\mathcal{O}(X)$ in $\mathrm{Lip}(\alpha,X)$ is the intersection of $\mathrm{lip}(\alpha,X)$ with the closure of $\mathcal{O}(X)$ in $\mathrm{Lip}(\alpha,E)$. But the polynomials in z and $1/z$ are dense in $\mathrm{lip}(\alpha,E)$, and since $z \neq 0$ on X, it follows that $\mathcal{O}(X)$ and hence the rationals in $C(X)$ are dense in $C(X)$.

More generally, let $X \subset \mathbb{C}^n$ be a compact holomorphically-convex set which is totally-real off a closed $\mathcal{O}(X)$-convex subset E, where E projects to area zero in each coordinate. For instance, take E with Hausdorff area zero. Then $\mathcal{O}(X)$ is dense in $\mathrm{lip}(\alpha,X)$, for $0 < \alpha < 1$. To show this, it suffices (in view of Theorem 1 and the lip α extension theorem) to show that the rationals are dense in $\mathrm{lip}(\alpha,F)$, where $F = \prod_{j=1}^{n} z_jE$ is the product of the coordinate projections of E. By the extended Hartogs-Rosenthal theorem [13, p. 287], the rationals in z_j are dense in $\mathrm{lip}(\alpha,z_jE)$, for each j. Thus the closure of the rationals in $\mathrm{Lip}(\alpha,F)$ contains the symmetric product $\otimes_j C^\infty(\mathbb{C})$, which is well-known to be dense in $C^\infty(\mathbb{C}^n)$. Since $C^\infty(\mathbb{C}^n)$ is dense in $\mathrm{lip}(\alpha,F)$, and $C^1(\mathbb{C}^n)$ convergence implies $\mathrm{lip}(\alpha,F)$ convergence, the result follows.

Obviously, area zero is the sharpest metric condition possible here, because positive area would allow possible analytic structure. The tricky point in applications is the $\mathcal{O}(X)$-convexity of E.

It seems plausible that the result should remain true in Lipschitzian submanifolds X in which the (no longer necessarily closed) set E where there are complex tangents has all coordinate projections of area zero.

(3.2) Let ρ be a C^2 strictly plurisubharmonic function on a neighborhood of the boundary of a compact set $X \subset \mathbb{C}^n$, with $\mathrm{bdy}\, X = \{\rho = 0\}$ and with $\{\rho < 0\} \subset D = \mathrm{int}\, X$. Then Theorems 1 and 2 apply, where $E = \mathrm{clos}\, D$.

Furthermore, $\mathcal{O}(X)$ is dense in $\mathcal{O}(E)$, in the usual Fréchet topology, and hence in $\mathrm{Lip}(\alpha,E)$ norm. This is seen as follows (cf. [8, Theorem 2.2(b)] for a similar argument). Choose a nonnegative C^∞ function φ on $\mathbb{C}^n$, bounded by 1, vanishing only on E. Then for all small constants $\varepsilon > 0$, the function $\rho + \varepsilon\varphi$ is strictly plurisubharmonic, vanishes only on bdy D and is negative only on D. Since $D \cup \{\rho + \varepsilon\varphi < 2\varepsilon\}$ contains $D \cup \{\rho < \varepsilon\}$, it is a (Stein) neighborhood of X. Call it U_2. It suffices to show that $\mathcal{O}(U_2)$ is dense in $\mathcal{O}(U_1)$, where $U_1 = D \cup \{\rho + \varepsilon\varphi < \eta\}$, for all sufficiently small positive constants $\eta \ll \varepsilon$. For this it suffices (by the functional calculus) to show that, given $K \subset U_1$ compact, the $\mathcal{O}(U_2)$-convex hull of K is a subset of U_1. Given such a K, choose a strictly plurisubharmonic exhaustion function u for U_2, with $u \le 0$ on K. Choose $a < 1$ such that

$$\rho + \varepsilon\varphi \le a\eta$$

on $K \sim D$, and let $L = \{u \le 0\} \cap \{\rho + \varepsilon\varphi \ge \eta\}$. Then $\rho + \varepsilon\varphi - a\eta \ge (1-a)\eta > 0$ on L, and L is compact, so there exists a constant $c > 0$ such that

$$\psi = u + c(\rho + \varepsilon\varphi - a\eta)$$

is positive on L. Clearly, ψ is a strictly plurisubharmonic exhaustion function for U_2, hence $\{u \le 0\} \cap \{\psi \le 0\}$ is $\mathcal{O}(U_2)$-convex. But

$$K \subset \{u \le 0\} \cap \{\psi \le 0\} \subset U_1,$$

so we are done. It follows that (1) and (2) hold. Thus, for such sets X, the approximation problems in Lip α and uniform norms are reduced to the problems on clos(int X).

Of course, Henkin and Leiterer [10, Lemma 3.5.4] have already established (2), and have gone on to show that

$$\mathrm{clos}_{C(X)}\, \mathcal{O}(X) = \{f \in C(X) : f \text{ is analytic on int } X\}.$$

This new proof of Lemma (3.5.4) is simpler. We hope to address the problem of proving that

$$\mathrm{clos}_{\mathrm{Lip}(\alpha,X)}\, \mathcal{O}(X) = \{f \in \mathrm{lip}(\alpha,X) : f \text{ is analytic in int } X\}$$

in a later paper.

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