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# APPROXIMATION BY POLYNOMIALS IN TWO DIFFEOMORPHISMS

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We denote by **C** the complex plane. If $f$ and $g$ are complex-valued functions on a set $S$, then $\mathbf{C}[f,g]$ denotes the algebra of polynomials in $f$ and $g$, with complex coefficients, regarded as functions on $S$.

THEOREM. *Let $1 \leq k \in Z$, and let $f$ and $g$ be $C^k$ diffeomorphisms of* **C** *into* **C**, *having opposite degrees. Then $\mathbf{C}[f,g]$ is dense in the Fréchet space $C^k(\mathbf{C})$, i.e., given $h \in C^k(\mathbf{C})$, and $X \subset \mathbf{C}$ compact, there is a sequence $h_n \in \mathbf{C}[f,g]$ such that $h_n$ and its derivatives up to order $k$ tend to $h$ and its derivatives, uniformly on $X$.*

In case $f(z) = z$ and $g(z) = \bar{z}$, the Theorem reduces to a result of Weierstrass.
Since each diffeomorphism of the closed unit disc $D$ into **C** extends to a diffeomorphism of **C** into **C**, we deduce the following.

COROLLARY. *Let $f$ and $g$ be $C^1$ diffeomorphisms of $D$ into* **C**, *having opposite degrees. Then $\mathbf{C}[f,g]$ is dense in $C(D)$.*

This settles an old chestnut in the field of uniform algebras. It remains open whether the Corollary works for $k = 0$, i.e., for all pairs of homeomorphisms of opposite degrees.

PROOF OF THEOREM. Without loss of generality, we may take $g = z$, because the chain rule for $D^j(h \circ g)$ is linear in $h$ and involves only $D^i h$ and $D^i g$ for $0 \leq i \leq j$.

Since $f$ has degree $-1$, we deduce that $|f_{\bar{z}}| > |f_z|$ on **C**. In particular, $f_{\bar{z}} \neq 0$, so the graph $G = \{(z, f(z)) \in \mathbf{C}^2 : z \in \mathbf{C}\}$, which is a $C^k$ submanifold of $\mathbf{C}^2$, has no complex tangents. By the Range-Siu theorem [**2**], $C^k(G)$ is the closure of the space $\mathcal{O}(G)$ of all functions holomorphic in a neighbourhood of $G$. If we can show that $G$ has an exhaustion by polynomially-convex compact sets, then by the functional calculus [**4**, Chapter 8], it will follow that $\mathbf{C}[z,w]$ is dense in $\mathcal{O}(G)$, and hence in $C^k(G)$; since $z \mapsto (z, f)$ is a $C^k$ diffeomorphism of $\mathbf{C} \to G$, this will imply that $\mathbf{C}[z,f]$ is dense in $C^k(\mathbf{C})$. Thus it suffices to show that $X = \{(z, f(z)) \colon z \in K\}$ is polynomially-convex whenever $K \subset \mathbf{C}$ is a closed disc.

Fix a closed disc $K \subset \mathbf{C}$. By modifying $f$ off $K$, if need be, we may assume $f$ maps **C** *onto* **C**, that $Df$ and $Df^{-1}$ are bounded and uniformly continuous, and that $|f_{\bar{z}}|$ and $1 - |f_z/f_{\bar{z}}|$ are bounded away from zero. We need two lemmas, which are essentially classical results of Wermer.

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LEMMA 1. *There exists a constant $\lambda_1 > 0$ such that*

$$(z-a)(f(z)-f(a))+\lambda f_{\bar z}(a)$$

*is nonzero whenever $0<\lambda<\lambda_1$, $a\in \mathbf{C}$, and $z \in \mathbf{C}$.*

PROOF. Pick $\delta > 0$ such that the modulus of continuity $\omega(\delta)$ of $Df$ at $\delta$ is less than half $(\inf |f_{\bar z}|)(1-\sup_{\mathbf{C}} |f_z/f_{\bar z}|$. Applying the mean value theorem to the real and imaginary parts of $f$ we deduce that for $0<|z-a|<\delta$, the value $f(z)-f(a)$ differs from $f_{\bar z}(a)\overline{(z-a)}+f_z(a)(z-a)$ by less than $2\omega(\delta)|z-a|$. Thus

$$\operatorname{Re}\frac{(z-a)(f(z)-f(a))}{f_{\bar z}(a)}\geq 0$$

whenever $|z-a|<\delta$. But for $|z-a|\geq\delta$,

$$\left|\frac{(z-a)(f(z)-f(a))}{f_{\bar z}(a)}\right|\geq\frac{\delta^2(\sup|Df^{-1}|)^{-1}}{\inf|f_{\bar z}|}.$$

Denoting the right-hand side by $\lambda_1$, we see that $(z-a)(f(z)-f(a))/f_{\bar z}(a)$ omits $\{-\lambda\colon 0<\lambda<\lambda_1\}$, for all $a$ and $z$, so the lemma is proved.

Let us denote the uniform closure of $\mathbf{C}[z,f]$ in $C(K)$ by $A$.

LEMMA 2. *Suppose that for each $a\in K$, there exists a sequence $\lambda_n\downarrow 0$ such that $(z-a)(f(z)-f(a))+\lambda_n f_{\bar z}(a)$ is invertible in $A$. Then $A=C(K)$.*

PROOF. Briefly, let $\mu$ be a measure on $K$, annihilating $A$. It suffices to show that the Cauchy transform $\hat\mu(a)=\int d\mu(\varsigma)/\varsigma-a$ vanishes at every point $a\in K$ at which the Newtonian potential $\int d|\mu|(\varsigma)/|\varsigma-a|$ is finite. But the hypothesis, together with Lemma 1, yields a sequence $f_n\in A$ such that $f_n\to (z-a)^{-1}$, pointwise on $K\sim\{a\}$, and $|f_n(z)|\leq$ const $|z-a|^{-1}$. Thus the dominated convergence theorem yields the desired result.

We remark that the hypothesis of Lemma 2 can be weakened to "almost all $a\in K$".

CONCLUSION OF PROOF OF THEOREM. Suppose $X$ is not polynomially-convex. Then $A\neq C(K)$, so by Lemma 2, there exists $a\in K$ and $\lambda_2>0$ such that for every $\lambda$ with $0<\lambda<\lambda_2$, the polynomial $(z-a)(w-f(a))+\lambda f_{\bar z}(a)$ has a zero somewhere on the polynomially-convex hull of $X$. Fix $\lambda$, with $0<\lambda<\min\{\lambda_1,\lambda_2\}$. Then the family of algebraic curves

$$(z-a-t)(w-f(a+t))+\lambda f_{\bar z}(a+t)=0 \qquad (0\leq t<\infty)$$

is a curve of algebraic hypersurfaces which meets the hull of $X$, does not meet $X$ (by Lemma 1), and goes to the hyperplane at infinity (since $f$ maps *onto* $\mathbf{C}$, and $f_{\bar z}$ is bounded). This contradicts Oka's characterization of polynomial hulls, as given in [3, (1.2), p. 263]. Thus $X$ is polynomially-convex, and we are done.

We remark that minor modifications to the foregoing proof permit us to strengthen the Corollary, as follows:

Let $f$ be an orientation-reversing homeomorphism of $\mathbf{C}$ into $\mathbf{C}$, which is locally $C^1$ and noncritical off a closed set $E$, having area zero and not separating the plane. Then $\mathbf{C}[z,f]$ is dense in $C(\mathbf{C})$.

Also, for any compact set $X$ in $\mathbf{C}$ and for $0 < \alpha < 1$, suppose $\mathrm{Lip}(\alpha, X)$ denotes the space of bounded functions $g$ of $X$ into $\mathbf{C}$ such that for some $K > 0$, $|g(z) - g(w)| \leq K|z-w|^\alpha$ for all $z, w \in X$ with norm $\sup|g| + \text{Least } K$ and suppose $\mathrm{lip}(\alpha, X)$ denotes those functions $g \in \mathrm{Lip}(\alpha, X)$ such that, given $\epsilon > 0$, there exists $\delta > 0$ such that $|g(z) - g(w)| \leq \epsilon|z-w|^\alpha$ whenever $z$ and $w$ satisfy $|z - w| < \delta$. In view of the results given in [**1**, p. 227], the conclusion of the above remark implies $\mathbf{C}[z, f]$ is dense in $\mathrm{lip}(\alpha, X)$ for any compact set $X$ in $\mathbf{C}$.

Finally, we remark that the Theorem of this paper is sharp in the sense that one critical point destroys it.

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