POLYNOMIAL APPROXIMATION OF SMOOTH FUNCTIONS

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1. *Introduction*

This paper is about the problem of how well an infinitely differentiable function can be approximated by polynomials. Let f be an infinitely differentiable function on a compact interval $I \subset \mathbb{R}$. The problem is to decide how 'small' the sequence $\|f^{(k)}-p^{(k)}\|_\infty$ $(k = 0, 1, 2, \ldots)$ can be made by choosing suitable polynomials p. Here $f^{(k)}$ stands for the k-th derivative of f, and so we are asking about simultaneous approximation of f and all its derivatives. It is well known that for fixed $n > 0$ and $\varepsilon > 0$ we can choose p such that $\|f^{(k)}-p^{(k)}\|_\infty < \varepsilon$ for $0 \leqslant k \leqslant n$. Thus the sequence can be made 'small' in the sense that the first n terms are small.

Evidently, for any polynomial p, the k-th derivative is zero for large k, and hence $\|f^{(k)}-p^{(k)}\|_\infty = \|f^{(k)}\|_\infty$. The numbers $\|f^{(k)}\|_\infty$ will usually be very large, and we have to bear this in mind in seeking an optimal notion of 'small'. Let $M = \{M_k\}$ be a sequence of positive numbers such that $M_k^{-1}\|f^{(k)}\|_\infty \to 0$. Given $\varepsilon > 0$, can we find polynomials p such that $\|f^{(k)}-p^{(k)}\|_\infty < \varepsilon M_k$ for all k?

We conjecture that the answer is yes, although it may, perhaps, be necessary to impose some modest regularity conditions on M. Prior to this, there have been some results [**3, 8, 11, 13, 19**] consistent with this conjecture. Most involve major regularity and growth restrictions on M, such as the following:

$$\text{for some fixed } a > 0, \qquad \frac{M_k}{M_j M_{k-j}} \geqslant a\binom{k}{j} \qquad \text{for } 0 \leqslant j \leqslant k\,; \tag{1}$$

$$\sum_{k=0}^{\infty}\left(\frac{k!}{M_k}\right)^{1/k} < +\infty\,. \tag{2}$$

The only unrestricted result is [**19**] that if $\lambda^k M_k^{-1}\|f^{(k)}\|_\infty \to 0$ for all $\lambda > 0$ as $k \uparrow \infty$, then there exists a sequence p_n of polynomials such that

$$\sup_k \lambda^k M_k^{-1}\|f^{(k)}-p_n^{(k)}\|_\infty \to 0$$

as $n \uparrow \infty$, for all $\lambda > 0$. The strongest result (communicated to the author by H. G. Dales) assumes both (1) and (2) and states that if

$$\sum_{k=1}^{\infty}\frac{\|f^{(k)}\|_\infty}{M_k} < +\infty\,,$$

then there exists a sequence of polynomials p_n such that

$$\sum_{k=1}^{\infty}\frac{\|f^{(k)}-p_n^{(k)}\|_\infty}{M_k} \to 0$$

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as $n \uparrow \infty$. We shall give a sharp result for a reasonably large class of sequences M.

We employ the notation

$$\mu_- = \liminf_{k \uparrow \infty} \left(\frac{M_k}{k!}\right)^{1/k}, \qquad \mu_+ = \limsup_{k \uparrow \infty} \left(\frac{M_k}{k!}\right)^{1/k}.$$

THEOREM 1. *Let I be a compact interval, let $M_k > 0$ for $k = 0, 1, 2, \ldots$, and let $f : I \to \mathbb{C}$ be such that $M_k^{-1}\|f^{(k)}\|_\infty \to 0$ as $k \uparrow \infty$. Let $M_k = 1$ for $k < 0$. Let $\varepsilon > 0$.*

(i) *If $\mu_- = +\infty$ and for some q*

$$\sum_{j=0}^{k} M_{j-q} M_{k-j-1}/M_k \text{ is bounded}, \tag{3}$$

then there exists a polynomial p such that $M_k^{-1}\|f^{(k)} - p^{(k)}\|_\infty < \varepsilon$ for all $k \geqslant 0$.

(ii) *If $\mu_- > 0$ and $\mu_+ < +\infty$, then there exists a polynomial p such that*

$$M_k^{-1}(\mu_+/\mu_-)^k \|f^{(k)} - p^{(k)}\|_\infty < \varepsilon$$

for all k.

We may call case (ii) the *analytic case*, because then f extends to an analytic function on the μ_+^{-1} neighbourhood of I.

The regularity condition (3) of the *non-analytic* case (i) is quite mild. It follows from (2), with $q = 2$. It also follows from (1). Indeed, it follows from the much weaker conditions that

$$M_k M_j \leqslant a M_{k+j}, \qquad M_{k+1} \geqslant b k^c M_k$$

where $a > 0$, $b > 0$ and $c > 0$ are independent of $k \geqslant 0$ and $j \geqslant 0$.

We shall obtain an analogous result for all L_p norms $1 \leqslant p < +\infty$, and other related results.

Before stating them, we remark that there are two respects in which Theorem 1 is deeper than classical results. First, it is a Banach space theorem rather than a nuclear space theorem. It is well known that questions about Banach spaces are more delicate. Secondly, it concerns a bounded interval. The analogous problems on the whole line and the circle involving approximation by entire functions and trigonometric polynomials are not as difficult, because the group structure is available (cf. **[13]**).

We proceed to define the Banach spaces of functions we shall use, and to state our main theorem.

Let $1 \leqslant p \leqslant \infty$ and $1 \leqslant r \leqslant \infty$. For a closed (possibly unbounded) interval $I \subset \mathbb{R}$, we define the space $\bar{B}(I, r, p, M)$ as the space of all infinitely differentiable functions $f : I \to \mathbb{C}$ such that the sequence $\|f^{(k)}\|_p$ of $L_p(I)$ norms belongs to the weighted sequence space $l_r(M_k^{-1})$, that is

$$\sum_{k=0}^{\infty} \{M_k^{-1}\|f^{(k)}\|_p\}^r < +\infty$$

if $1 < r < +\infty$, and

$$\sup_k M_k^{-1}\|f^{(k)}\| < +\infty$$

2. What is the analogue of the Rudin–Carleson interpolation theorem for the analytic functions on the disc, of class B up to the boundary? The corresponding problem for the nuclear Gevrey classes has been solved **[14]**.

3. The fact that each space $\bar{B} = \bar{B}(I, r, p, M)$ is complemented in its second dual $\bar{B}^{**}$ suggests that $\bar{B}$ might be a dual space. Is this true? For $1 < r < \infty$ and $1 < p < \infty$, $\bar{B}$ is reflexive, and so the question really concerns the remaining cases.

2. *The second dual and dominated convergence*

Let M be a fixed sequence of positive numbers, let $1 \leqslant r \leqslant \infty$, $1 \leqslant p \leqslant \infty$, and let I be a closed interval. The map

$$f \longmapsto (f, f', f'', \ldots, f^{(k)}, \ldots)$$

imbeds $\bar{B} = \bar{B}(I, r, p, M)$ isometrically in the Banach space

$$A = \bigoplus_{k=0}^{\infty} L_p(I),$$

where the norm is the $l_r(M_k^{-1})$ norm of the L_p norms on I. If $1 < r < \infty$ and $1 < p < \infty$, then A is reflexive, and hence so is $\bar{B}$. Our first objective is to show that in all cases there is an explicit norm 1 projection from $\bar{B}^{**}$ onto $\bar{B}$. We shall then use this to establish a dominated convergence principle for $\bar{B}$.

The elementary estimate

$$|f(x)| \leqslant \frac{2}{\alpha}\|f\|_p + \frac{\alpha}{2}\|f'\|_p \qquad (x \in I,\ 0 < \alpha < |I|) \tag{1}$$

shows that the linear functionals $\delta^k(x) : f \mapsto f^{(k)}(x)$ are well defined and continuous on $\bar{B}$. Thus, given $F \in \bar{B}^{**}$, we may define associated functions $F_k : I \to \mathbb{C}$ by

$$F_k(x) = F(\delta^k(x)).$$

(2.1) LEMMA. *The linear map $P : F \to F_0$ is a norm 1 projection from $\bar{B}^{**}$ onto $\bar{B}$.*

Proof. By 'projection' we mean, of course, a map which inverts the natural injection of $\bar{B}$ into $\bar{B}^{**}$. This injection Q is defined by $(Qf)(L) = Lf$ for $L \in \bar{B}^{*}$, and so evidently $PQf = f$.

We may assume that $[-1, 1] \subset I$, without loss of generality. Fix $F \in \bar{B}^{**}$, with $\|F\| = 1$. We have to show that $F_0 \in \bar{B}$, and $\|F_0\| \leqslant 1$. For $k \geqslant 0$ and $f \in \bar{B}$,

$$\begin{aligned}\left|\left(\frac{\delta^k(x+h) - \delta^k(x)}{h} - \delta^{k+1}(x)\right) f\right| &= \left|\frac{f^{(k)}(x+h) - f^{(k)}(x)}{h} - f^{(k+1)}(x)\right| \\ &= |\tfrac{1}{2} h f^{(k+2)}(x + \theta h)|, \qquad \text{for some } \theta,\ 0 < \theta < 1, \\ &\leqslant |h| \|f^{(k+2)}\|_\infty \leqslant |h| \{\|f^{(k+2)}\|_p + \|f^{(k+3)}\|_p\} \\ &\leqslant |h| \max\{M_{k+2}, M_{k+3}\} \|f\|_B;\end{aligned}$$

hence

$$\lim \frac{\delta^k(x+h)+\delta^k(x)}{h} = \delta^{k+1}(x),$$

where the limit is taken in the norm of $\bar{B}^*$. Since F is continuous on $\bar{B}^*$, we get $F_k' = F_{k+1}$ on I. We conclude that $F_0^{(k)} = F_k$, and that F_k is continuous, and hence measurable.

Let s and q be the conjugate indices to r and p, respectively.

Case 1^0, *in which* $1 \leqslant r < \infty$. If $\|F_0\|_B > 1$, then there exists a sequence $\alpha_k \in \mathbb{R}$ such that

$$\sum_k (M_k|\alpha_k|)^s = 1 \qquad \text{and} \qquad \sum_k \alpha_k \|F_k\|_p > 1 .$$

We may assume that $\alpha_k = 0$ for all but a finite number of indices k. We can find continuous functions $g_k \in L_q(I)$ such that $\|g_k\|_q = 1$ and

$$\sum_k \alpha_k \int_{-1}^{1} F_k g_k dx > 1 .$$

(Consider separately the cases when $1 \leqslant p < \infty$ and $p = \infty$.) The linear functional

$$L : f \longrightarrow \sum_k \alpha_k \int_{-1}^{1} f^{(k)} g_k dx$$

is continuous on $\bar{B}$, and has norm at most 1. Using the continuity of the g_k and F we see easily that

$$F(L) = \sum_k \alpha_k \int_{-1}^{1} F_k g_k dx .$$

But this means that $\|F\|$ exceeds 1, which is impossible. We conclude that $\|F_0\|_B \leqslant 1$.

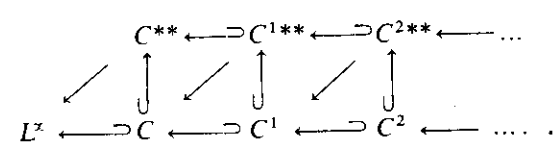
Case 2^0, *in which* $r = \infty$. If $\|F_0\|_B > 1$, then there exists k such that $\|F_k\|_p > M_k$. We can find a smooth function g on I such that $\|g\|_q = 1$ and $\int_{-1}^{1} F_k g\, dx > M_k$. Much as before, we obtain the contradiction that $\|F\| > 1$. Thus $\|F_0\|_B \leqslant 1$.

This concludes the proof. For future reference, we record the formula

$$F_0^{(k)} = F_k \tag{4}$$

which holds for $F \in \bar{B}^{**}$.

The above theorem may be viewed as a limit of the diagram



The diagonal projection, from C^{k+1**} into C^k, is defined by the same rule $F \to F_0$ as in the argument above. The image of this projection is $C^k \cap \{f : f^{(k)} \in \text{Lip}\, 1\}$.

Now we formulate and prove the dominated convergence principle.

Let $f_n \in B(I, r, p, M)$, $n = 1, 2, 3, \ldots$, and let $\beta_k \geqslant 0$, $k = 0, 1, 2, \ldots$. We say that $\{f_n\}$ is *(I, r, p, M)-dominated* by $\{\beta_k\}$ if

(1) $1 \leqslant r < \infty$ and $\displaystyle\sum_{k=0}^{\infty} \left(\frac{\beta_k}{M_k}\right)^r < \infty$, *or* $r = \infty$ and $\dfrac{\beta_k}{M_k} \to 0$;

(2) $\|f_n^{(k)}\|_p \leqslant \beta_k$ for all k and n.

(2.2) LEMMA. *Let I be a closed bounded interval. Let $f_n, f \in \bar{B}$, and suppose that $\|f_n\| \leqslant 1$ and $f_n(x) \to f(x)$ for each $x \in I$. Then $\|f_n^{(k)} - f^{(k)}\|_p \to 0$ for each k.*

Proof. Let F be any weak-star accumulation point of $\{f_n\}$ in $\bar{B}^{**}$. Then, with the notation of the last proof, $F_0(x) = f(x)$ for all $x \in I$. Thus $F_k(x) = f^{(k)}(x)$, for all x. Since $F_k(x) = F(\delta^k(x))$, this implies that $f_n^{(k)}(x) \to f^{(k)}(x)$ for each x. For $1 \leqslant p < \infty$, the Lebesgue dominated convergence theorem implies that $\|f_n^{(k)} - f^{(k)}\|_p \to 0$. For $p = \infty$, we obtain the same conclusion from the fact that $f_n^{(k+1)} \to f^{(k+1)}$ in L_1 norm.

(2.3) THEOREM. *Let I be a closed bounded interval, let $f_n, f \in B(I, r, p, M)$. Suppose that $f_n(x) \to f(x)$ for each $x \in I$, and $\{f_n\}$ is (I, r, p, M)-dominated by a sequence $\{\beta_k\}$. Then $f_n \to f$ in the norm of B.*

Proof. By the lemma, $f_n^{(k)} \to f^{(k)}$ in L_p for each k. Given $\varepsilon > 0$, pick K such that

$$\sum_{k=K+1}^{\infty} \left(\frac{\beta_k}{M_k}\right)^r < \frac{\varepsilon}{3} \qquad (\text{when } 1 \leqslant r < \infty),$$

$$\sup_{k > K} \frac{\beta_k}{M_k} < \frac{\varepsilon}{3} \qquad (\text{when } r = \infty).$$

Then pick N such that

$$\|f_n^{(k)} - f^{(k)}\|_p < \frac{\varepsilon M_k}{3(K+1)} \qquad (\text{when } 1 \leqslant r < \infty),$$

$$\|f_n^{(k)} - f^{(k)}\|_p < \varepsilon M_k \qquad (\text{when } r = \infty)$$

wherever $0 \leqslant k \leqslant K$ and $n > N$. Then $\|f_n - f\|_B < \varepsilon$ whenever $n > N$. This proves the result.

We conclude this section by giving the corresponding theorem on the whole line $\mathbb{R}$, and an application.

(2.4) THEOREM. *Let $f_n, f \in B(\mathbb{R}, r, p, M)$, and suppose that $f_n \to f$ in L_p norm and $\{f_n\}$ is $(\mathbb{R}, r, p, M)$-dominated by a sequence $\{\beta_k\}$. Then $f_n \to f$ in $B(\mathbb{R}, r, p, M)$ norm.*

It is no longer enough to assume merely pointwise convergence. This result is proved in the same way as the last one.

(2.5) COROLLARY. *The entire functions are dense in* $B(\mathbb{R}, r, p, M)$.

Proof. Fix $f \in B(\mathbb{R}, r, p, M)$. Form the convolution $f_n = f * \phi_n$, where

$$\phi_n(t) = (2\pi n)^{-1/2} e^{-x^2/n}$$

is the Weierstrass kernel. Then $\|f_n^{(k)}\|_p \leqslant \|f^{(k)}\|_p$ and $f_n \to f$ in $L_p(\mathbb{R})$. By the theorem, $f_n \to f$ in B norm. Clearly, f_n is entire, and so we are done.

This result is the starting point of Dales' proof of the main theorem for the *algebra* $B(I, 1, \infty, M)$ in the case of M satisfying (1) and (2). In that case any $f \in B$ has an extension in $B(\mathbb{R}, 1, \infty, M)$. By the corollary, f is a limit of entire functions in $B(I, 1, \infty, M)$. Since entire functions act on any Banach algebra, each entire function is a limit of polynomials in $B(I, 1, \infty, M)$ norm.

3. *Reductions*

Let $M_k > 0$ for $k \in \mathbb{Z}$, and $M_k = 1$ for $k < 0$. Note that each of the conditions (1), (2), (3), $\mu_- = \infty$, $\mu_- > 0$, $\mu_+ < +\infty$, $\mu_- = \mu_+$ is such that, if it holds for M, then it also holds for the sequences $\alpha^k M_k$ and $N_k = \min\{M_k, M_{k-1}\}$. Also, the map taking f on $[a, b]$ to the function

$$x \longmapsto f\left(a_1 + \left(\frac{b-a}{b_1 - a_1}\right)(x - a_1)\right)$$

gives a linear isometry from $B([a, b], r, p, M)$ onto $B([a_1, b_1], r, p, \alpha^k M_k)$, where $\alpha = \dfrac{b-a}{b_1 - a_1}$. This usually allows us to work with $I = [-1, 1]$, without loss of generality.

(3.1) LEMMA. *The set* $\bigcup_{\alpha > 1} B(I, r, p, \alpha^{-k} M_k)$ *is dense in* $B(I, r, p, M)$.

Proof. It suffices to take $I = [-1, 1]$. For $\alpha > 1$,

$$\left\|\frac{d^k}{dx^k} f(\alpha^{-1} x)\right\|_\infty \leqslant \alpha^{-k} \|f^{(k)}\|_p \leqslant \|f^{(k)}\|_p .$$

Since $f(\alpha^{-1} x) \to f(x)$ pointwise as $\alpha \downarrow 1$, the dominated convergence principle shows that $f(\alpha^{-1} x) \to f(x)$ in $B(I, r, p, M)$ norm.

(3.2) LEMMA. *Let* $N_k = \min\{M_{k-1}, M_k\}$. *Then* $B(I, r, p, N)$ *is dense in* $B(I, r, p, M)$.

Proof. It suffices to consider $I = [-1, 1]$. We abbreviate $B(I, r, p, M)$ to $B(M)$.

For $0 < \varepsilon < \frac{1}{2}$, the linear operator T_ε defined by

$$(T_\varepsilon f)(x) = \frac{1}{2\varepsilon} \int_{x-\varepsilon x-\varepsilon}^{x-\varepsilon x+\varepsilon} f(t)dt$$

maps $L_p(I)$ continuously to $L_p(I)$. To show this, it suffices, by the Marcinkiewicz interpolation theorem, to show that T_ε maps L_∞ to L_∞ and L_1 to L_1. It is obvious that

$$\|T_\varepsilon f\|_\infty \leqslant \|f\|_\infty .$$

As for L_1, we have

$$2\varepsilon \int_{-1}^{1} |T_\varepsilon f(x)| dx \leqslant \int_{-1}^{1} \int_{-1}^{1} \chi(x, t)|f(t)| dt dx ,$$

where

$$\chi(x, t) = \begin{cases} 1 & \text{if } x-\varepsilon x-\varepsilon < t < x-\varepsilon x+\varepsilon , \\ 0 & \text{otherwise} . \end{cases}$$

Thus, by Fubini's theorem,

$$\|T_\varepsilon f\|_1 \leqslant (1-\varepsilon)^{-1}\|f\|_1 .$$

We conclude that for some $A_p > 0$, $\|T_\varepsilon f\|_p \leqslant A_p\|f\|_p$.

Next, note that for $f \in B(M)$,

$$(T_\varepsilon f)^{(k+1)}(x) = \frac{(1-\varepsilon)^{k+1}}{2\varepsilon} \{f^{(k)}(x-\varepsilon x+\varepsilon) - f^{(k)}(x-\varepsilon x-\varepsilon)\}$$

$$= (1-\varepsilon)^{k+1}(T_\varepsilon f^{(k+1)})(x) .$$

This shows that $T_\varepsilon f$ belongs to $B(M_{k-1})$ and to $B(M)$, and hence to $B(N)$. Furthermore,

$$\|(T_\varepsilon f)^{(k+1)}\|_p \leqslant \|T_\varepsilon f^{(k+1)}\|_p \leqslant A_p\|f^{(k+1)}\|$$

and $T_\varepsilon f \to f$ pointwise as $\varepsilon \downarrow 0$, so that, by dominated convergence, $T_\varepsilon f \to f$ in $B(M)$ norm.

(3.3) LEMMA. *Let N be as above, and suppose that for some r and p the polynomials are dense in $B(I, r, p, M)$, $B(I, r, p, N)$, and $B(I, r, p, \alpha^{-k}M_k)$, for all $\alpha > 1$. Then they are dense in $B(I, r, p, M)$ for all r and p.*

Proof. Let $1 \leqslant p' \leqslant p \leqslant p'' \leqslant +\infty$. Then we have continuous inclusions

$$B(I, r, p, N) \subset B(I, r, p', N) \subset B(I, r, 1, N) \subset B(I, r, \infty, M)$$

$$\subset B(I, r, p'', M) \subset B(I, r, p, M) .$$

By (3.2), $B(I, r, p, N)$ is dense in $B(I, r, p, M)$; hence all terms of the chain are dense in $B(I,r,p,M)$. Thus if the polynomials are dense in $B(I,r,p',N)$ or in $B(I,r,p'',M)$, then they are dense in $B(I,r,p,M)$.

We also have the chain

$$B(I,r,p,\alpha^{-k}M_k) \subset B(I,r'',p,\alpha^{-k}M_k) \subset B(I,1,p,M)$$
$$\subset B(I,r',p,M) \subset B(I,r,p,M),$$

valid wherever $1 \leqslant r' \leqslant r \leqslant r'' \leqslant +\infty$. In view of (3.1), we conclude that if the polynomials are dense in $B(I,r'',p,\alpha^{-k}M_k)$ for all $\alpha > 1$, or are dense in $B(I,r',p,M)$, then they are dense in $B(I,r,p,M)$. The result follows.

4. *Proof of the main theorem*

In this section we prove Theorem 2. In view of the results of §3, it suffices to consider $B(I,1,\infty,M)$ or $B(I,\infty,\infty,M)$. Fix a closed bounded interval I.

First, we prove the analytic case, writing $B(M)$ for $B(I,1,\infty,M)$. Let

$$0 < \mu_- = \liminf_{k\uparrow\infty} (M_k/k!)^{1/k},$$

$$\limsup_{k\uparrow\infty} (M_k/k!)^{1/k} = \mu_+ < +\infty,$$

$$\beta = \mu_+/\mu_-.$$

Fix $f \in B(M)$ and $\alpha > 1$. Then $f(\alpha^{-1}x)$ extends to an analytic function f_α on the set

$$U = \{z \in \mathbb{C} : \operatorname{dist}(z,I) < \alpha\mu_+^{-1}\}.$$

Now $B(\{\mu_+^k k!\})$ is a Banach algebra, in which the spectrum of the function $x \mapsto x$ is

$$\{z \in \mathbb{C} : \operatorname{dist}(z,I) \leqslant \mu_+^{-1}\}.$$

By the functional calculus for Banach algebras, f_α is a limit of polynomials in $B(\{\mu_+^k k!\})$ norm. There exists $A_1 > 0$ such that $k! \leqslant A_1\mu_-^{-k}M_k$ for all k, and hence $\mu_+^k k! \leqslant A_1\beta^k M_k$. Thus f_α is a limit of polynomials in $B(\{\beta^k M_k\})$ norm. Since $f_\alpha \to f$ in $B(M)$ norm as $\alpha \downarrow 1$, we are done.

Now consider the non-analytic case, that is, that where (3) holds and

$$\lim_{k\uparrow\infty} (M_k/k!)^{1/k} = +\infty.$$

This time we use $B(I,M)$ for $B(I,\infty,\infty,M)$. Let N_k denote the minimum of $M_k, M_{k-1}, \ldots, M_{k-q}$. Let $I = [-1,1]$. By §3, it suffices to show that if $\alpha > 1$ and $f \in B([-\alpha,\alpha],N)$, then f is a limit of polynomials in $B(I,M)$ norm. Since $\beta^k k!/M_k$ is bounded for each $\beta > 0$, and the analytic case is proved, it suffices to prove that f is a limit in $B(I,M)$ norm of functions analytic in a neighbourhood of $[-1,1]$.

Consider the (truncated) Poisson integral

$$u(x,y) = \int_{-\alpha}^{\alpha} f(s)Py(x-s)\,ds = \int_{x-\alpha}^{x+\alpha} f(x-t)P_y(t)\,dt,$$

where $P_y(t) = \dfrac{1}{\pi}\cdot\dfrac{t}{t^2+y^2}$. As $y \downarrow 0$, the function $x \mapsto u(x, y)$ tends pointwise to f on I. Also,

$$\frac{\partial^k u}{\partial x^k} = \sum_{j=0}^{k-1} \{f^{(j)}(-\alpha)P_y^{(k-j-1)}(x+\alpha) - f^{(j)}(\alpha)P_y^{(k-j-1)}(x-\alpha)\} + \int_{-\alpha}^{\alpha} f^{(k)}(s)P_y(x-s)\,ds\,,$$

$$P_y^{(k)}(t) = \frac{(-1)^k k!}{2\pi}\{(t-iy)^{-k-1} + (t+iy)^{-k-1}\}\,.$$

Let $\|f\|_\infty$ denote, for the moment, $\|f\|_{L_\infty(-\alpha,\alpha)}$. If $x \in [-1, 1]$, then $|x \pm \alpha| \geqslant \alpha - 1 = \beta$, say, and hence

$$|P_y^{(k)}(x \pm \alpha)| \leqslant k!\,\pi^{-1}\beta^{-k-1}\,,$$

$$\left|\frac{\partial^k u}{\partial x^k}(x, y)\right| \leqslant 2\pi^{-1}\sum_{j=0}^{k-1} \|f^{(j)}\|_\infty (k-j-1)!\,\beta^{-k+j} + \|f^{(k)}\|_\infty\,.$$

There exist $A_2 > 0$ and $A_3 > 0$, independent of k, such that

$$k!\,\beta^{-k} \leqslant A_2 M_k\,, \qquad \|f^{(k)}\|_\infty \leqslant A_3 N_k \leqslant A_3 M_{k-q}\,.$$

Thus

$$\left\|\frac{\partial^k u}{\partial x^k}\right\|_{L_\infty(I)} \leqslant 2\pi^{-1}\beta^{-1}A_2A_3\sum_{j=0}^{k-1} M_{j-q}M_{k-j-1} + A_3M_{k-q}\,.$$

By condition (3) and dominated convergence, $u(x, y) \mapsto f(x)$ in $B(I, M)$ norm as $y \downarrow 0$.

This concludes the proof.

References

1. T. Bang, 'On quasi-analytic functions', *Math. Scand.*, 1 (1953), 137–152.
2. E. Borel, *Leçons sur les fonctions monogènes uniformes* (Gauthier–Villars, Paris, 1917).
3. J. Bruna, 'Sobre un algebra de funciones casi analiticas', *Collect. Math.*, 30 (1979), 113–123.
4. J. Bruna, 'An extension theorem of Whitney type for non quasi-analytic classes of functions', *J. London Math. Soc.* (2), 22 (1980), 495–505.
5. J. Bruna, 'Les algebras de Denjoy–Carleman analiticas', *Collect. Math.*, 30 (1979), 213–225.
6. J. Bruna, 'Spectral synthesis in non quasi-analytic classes of infinitely differentiable functions', *Bull. Sci. Math.* (2), 104 (1980), 65–78.
7. J. Bruna, 'On inverse-closed algebras of infinitely-differentiable functions', *Studia Math.*, 69 (1980), 59–68.
8. T. Carleman, *Les fonctions quasi analytiques* (Gauthier–Villars, Paris, 1926).
9. L. Carleson, 'On universal moment problems', *Math. Scand.*, 9 (1961), 197–206.
10. P. J. Cohen, 'A simple proof of the Denjoy–Carleman theorem', *Amer. Math. Monthly*, 75 (1968), 26–31.
11. C. C. Chou, *La transformation de Fourier complexe et l'equation de convolution*, Lecture Notes in Mathematics 325 (Springer, Berlin, 1973).
12. H. G. Dales and A. M. Davie, 'Quasianalytic Banach function algebras', *J. Funct. Anal.*, 13 (1973), 28–50.
13. H. G. Dales and J. P. McClure, 'Completion of normed algebras of polynomials', *J. Austral. Math. Soc.*, 20 (1975), 504–510.

14. E. M. DYNKIN and S. V. HRUSHCHEV, 'Interpolation by analytic functions, smooth up to the boundary', *Zap. Naucn. Sem. LOMI*, 56 (1976) 59–72.
15. G. A. DZHANASHIYA, 'Carleman's problem for the class of Gevrey functions', *Soviet Math.*, 3 (1962), 969–972.
16. L. EHRENPREIS, *The punctual and local images of quasianalytic and nonquasianalytic classes* (Institute for Advanced Study, Princeton, 1961).
17. S. MANDELBROJT, *Series adhérentes, régularisation des suites, applications* (Gauthier–Villars, Paris, 1952).
18. B. S. MITYAGIN, 'An infinitely differentiable function with the values of its derivatives given at a point', *Soviet Math.*, 2 (1961), 594–597.
19. A. G. O'FARRELL, 'The Legendre expansion of smooth function', preprint, Maynooth College, 1982.
20. B. A. TAYLOR, 'Analytically uniform spaces of infinitely-differentiable functions', *Comm. Pure Appl. Math.*, 24 (1971), 39–51.
21. G. WAHDE, 'Interpolation in nonquasianalytic classes of infinitely-differentiable functions', *Math. Scand.*, 20 (1967), 19–31.

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