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Polynomial Approximation on Graphs

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1. Introduction

This paper is about approximation by (analytic) polynomials in $2n$ complex variables on the graph $X: w=f(z)$ of a function $f: Y \to \mathbb{C}^n$, where $Y \subset \mathbb{C}^n$ is compact. The result is as follows (see Sect. 2 for notation).

Theorem. *Let $Y \subset \mathbb{C}^n$ be compact. Let f be a* Lip 1 *function on a neighbourhood of Y, with values in $\mathbb{C}^n$. Assume that $X = \{(z, f(z)) : z \in Y\}$ is a polynomially-convex subset of $\mathbb{C}^{2n}$. Let $E \subset Y$ be closed, and suppose that at each point $a \in Y \sim E$, the Frechet derivative $Df(a)$ exists and is continuous, and the matrix $f_{\bar{z}}(a)$ of $\bar{z}$-derivatives is nonsingular. Let $0 < \alpha < 1$. Denote by $\tilde{E}$ the set $\{(z, f(z)) : z \in E\}$. Then a function $g: X \to \mathbb{C}$ is a limit of polynomials in* $\mathrm{Lip}(\alpha, X)$ *norm, if and only if $g \in \mathrm{lip}(\alpha, X)$, and the restriction $g|\tilde{E}$ is a limit of polynomials in* $\mathrm{Lip}(\alpha, \tilde{E})$ *norm, i.e.*

$$\mathrm{clos}_{\mathrm{Lip}(\alpha, X)} \mathbb{C}[z, w] = \mathrm{lip}(\alpha, X) \cap \mathrm{clos}_{\mathrm{Lip}(\alpha, \tilde{E})} \mathbb{C}[z, w].$$

For example, if f is continuously-differentiable on a neighbourhood of Y, and the polynomials in z and f are dense in $\mathrm{lip}(\alpha, E)$, where $E = \{a \in Y : \det f_{\bar{z}}(a) = 0\}$, then the polynomials in z and w are dense in $\mathrm{lip}(\alpha, X)$.

The case $E = \emptyset$ of the theorem follows from a result of Range and Siu, at least in the case when Y is diffeomorphic to a ball. They proved [8] that, in that case, the functions analytic on a neighbourhood of X are dense in $C^1(X)$ (even without the assumption of polynomial convexity). Since $C^1(X)$ convergence implies $\mathrm{Lip}(\alpha, X)$ convergence, and $C^1(X)$ is dense in $\mathrm{lip}(\alpha, X)$, the case $E = \emptyset$ of our theorem follows, by the functional calculus for Banach algebras [2, III.4)], in view of the polynomial convexity of X.

An analogous result for uniform approximation was proved by Weinstock [14].

We prove the theorem by using integral transform of distributions. We combine the ideas of Weinstock, based on Cauchy-Fantappié kernels, with the ideas of [5, 7]. The resulting proof is simpler, in the case $E = \emptyset$, than that via the Range-Siu result.

2. Notation

For a Banach space B, and a subset $A\subset B$, we denote the closure of A in B by $\operatorname{clos}_B A$. We denote the dual space of B by B^*. We denote the Frechet space of C^∞ functions $f:\mathbb{C}^n\to\mathbb{C}$, having compact support in an open set $U\subset\mathbb{C}^n$, by $\mathscr{D}(U)$. For a compact set $X\subset\mathbb{C}^n$ and $0<\alpha\leqq 1$, $\operatorname{Lip}(\alpha,X)$ denotes the Banach algebra of all functions $f:X\to\mathbb{C}$ such that for some $\kappa>0$,

$$|f(x)-f(y)|\leqq\kappa|x-y|^\alpha$$

whenever $x, y\in X$; the norm of such an f is the sum $\|f\|_\infty+\|f\|'_\alpha$, where $\|f\|'_\alpha$ denotes the least value of κ. The closed subalgebra $\operatorname{lip}(\alpha,X)$ consists of those f such that, for each $\in>0$, there exists $\delta>0$ such that

$$|f(x)-f(y)|\leqq\in|x-y|^\alpha$$

whenever $|x-y|<\delta$. The space $\mathbb{C}[z,w]$ consists of all polynomial $\mathbb{C}$-valued functions of $z=(z_1,\ldots,z_n)$ and $w=(w_1,\ldots,w_n)$, with complex coefficients. The notation $\langle z,w\rangle$ stands for $z_1w_1+\ldots+z_nw_n$; note that it is bilinear, *not* sesquilinear. We denote n-dimensional Lebesgue measure by $\mathscr{L}^n$, and the total variation measure of a complex measure μ by $|\mu|$, i.e. for each Borel set E,

$$|\mu|E=\sup\sum_{n=1}^{\infty}|\mu E_n|,$$

where $\{E_n\}_1^\infty$ runs over all countable partitions of E into Borel sets.

3. Proof of Theorem

The assertion we wish to prove is equivalent to

$$\operatorname{clos}_{\operatorname{Lip}(\alpha,Y)}\mathbb{C}[z,f(z)]=\operatorname{lip}(\alpha,Y)\cap\operatorname{clos}_{\operatorname{Lip}(\alpha,E)}\mathbb{C}[z,f(z)]. \tag{1}$$

This equivalence rests on the fact that the map

$$\begin{gathered}Y\to X\\ z\mapsto(z,f(z))\end{gathered}$$

is biLipschitzian.

It is straightforward that the left-hand side of (1) is a subset of the right-hand side.

To prove the converse, it suffices, in view of the separation theorem, to show that if a continuous linear functional $T\in\operatorname{Lip}(\alpha,Y)^*$ annihilates $\mathbb{C}[z,f(z)]$, then T also annihilates the right-hand side in (1). Let $T_1=T|\mathscr{D}(\mathbb{C}^n)$ be the distribution induced by T. Suppose we can show that T_1 is supported on E. Let $g\in\operatorname{lip}(\alpha,Y)\cap\operatorname{clos}_{\operatorname{Lip}(\alpha,E)}\mathbb{C}[z,f]$. Choose a sequence $p_m\in\mathbb{C}[z,f]$ such that $p_m\to g$ in $\operatorname{Lip}(\alpha,E)$ norm. There is a continuous extension operator from $\operatorname{lip}(\alpha,E)$ to $\operatorname{lip}(\alpha,Y)$ [13, (VI.2.2.3), p. 175], so there exist functions $g_m\in\operatorname{lip}(\alpha,Y)$ such that $g_m=g-p_m$ on E and $g_m\to 0$ in $\operatorname{Lip}(\alpha,Y)$ norm. By [11, 4], each function belonging to $\operatorname{lip}(\alpha,Y)$ which vanishes on E is the limit in $\operatorname{Lip}(\alpha,Y)$ norm of functions which vanish on a neighbourhood of E. Thus there exist functions $g_{mr} \in \operatorname{lip}(\alpha, Y)$ $(r = 1, 2, 3, \ldots)$ such that g_{mr} vanishes on a neighbourhood of E and $g_{mr} \to g - p_m - g_m$ in $\operatorname{Lip}(\alpha, Y)$ norm, as $r \uparrow \infty$. By extending g_{mr} to a lipα function on $\mathbb{C}^n$, having compact support, and by convolving with a sequence of mollifiers, we obtain a sequence g_{mrs} $(s = 1, 2, 3, \ldots)$ of elements of $\mathscr{D}(\mathbb{C}^n)$, vanishing on a neighbourhood of E, such that $g_{mrs} \to g_{mr}$ in $\operatorname{Lip}(\alpha, Y)$ norm, as $s \uparrow \infty$. Thus $T g_{mr} = \lim_{s \uparrow \infty} T_1 g_{mrs} = 0$. Thus

$$\begin{aligned} Tg &= T(g - p_m) = T(g - p_m - g_m) + T g_m \\ &= \lim_{r \uparrow \infty} T g_{mr} + T g_m = T g_m \to 0 \end{aligned}$$

as $m \uparrow \infty$, hence $Tg = 0$. Thus it suffices to show that T_1 is supported on E.

To show that T_1 is supported on E, it suffices to show that each point $a \in Y \sim E$ has a neighbourhood U such that $T\varphi = 0$ whenever $\varphi \in \mathscr{D}(U)$.

Fix $a \in Y \sim E$. We will show that there exist a neighbourhood U of a, a neighbourhood V of Y, and a function $\Omega(\zeta, z)$ mapping each $z \in V$ to a $(2n-1)$ form of type $(n, n-1)$ in ζ on U, such that

$$\varphi(z) = \int_{\zeta \in U} \Omega(\zeta, z) \wedge \bar{\partial} \varphi(\zeta)$$

whenever $z \in V$ and $\varphi \in \mathscr{D}(U)$. The form

$$\Omega = \sum_{j=1}^{n} K_j(\zeta, z) d\bar{\zeta}_1 \wedge \ldots \wedge d\bar{\zeta}_{j-1} \wedge d\bar{\zeta}_{j+1} \wedge \ldots \wedge d\bar{\zeta}_n \wedge d\zeta_1 \wedge \ldots \wedge d\zeta_n$$

will have certain additional properties, specified below.

There exists [5, p. 386] a complex measure μ on $Y \times Y$, having no mass on the diagonal, such that

$$Tg = \int_{Y \times Y} \frac{g(x) - g(y)}{|x - y|^{\alpha}} \, d\mu(x, y)$$

whenever $g \in \operatorname{lip}(\alpha, Y)$. Thus, for $\varphi \in \mathscr{D}(U)$, we have $T\varphi$ represented as

$$\int_{Y \times Y} \int_{U} \{\Omega(\zeta, x) - \Omega(\zeta, y)\} \wedge \bar{\partial} \varphi(\zeta) \frac{d\mu(x, y)}{|x - y|^{\alpha}}.$$

The additional properties of Ω, referred to above, are as follows. First, there exists a constant $M_1 > 0$ such that

$$\int_{\zeta \in U} |\{\Omega(\zeta, x) - \Omega(\zeta, y)\} \wedge \omega(\zeta)| \leqq M_1 |x - y|^{\alpha} \|\omega\|_{\infty} \tag{2}$$

for all $x, y \in Y$ and all $(0, 1)$ forms ω on U with bounded measurable coefficients. This allows us to apply Fubini's theorem to write $T\varphi$ as

$$\int_{U} \int_{Y \times Y} \{\Omega(\zeta, x) - \Omega(\zeta, y)\} \frac{d\mu(x, y)}{|x - y|^{\alpha}} \wedge \bar{\partial} \varphi(\zeta).$$

The second property of Ω is that for $\mathscr{L}^{2n}$ almost all $\zeta \in U$ and each $j \in \{1, \ldots, n\}$ there exists a positive function $\Phi(\zeta, x, y)$ on $Y \times Y$ such that

$$\int_{Y \times Y} \Phi(\zeta, x, y) \frac{d\mu(x, y)}{|x-y|^\alpha} < +\infty \tag{3}$$

and there exists a sequence $p_m(z, w) \in \mathbb{C}[z, w]$ such that $p_m(z, f(z)) \to K_j(\zeta, z)$ pointwise on $Y \sim \{\zeta\}$ as $m \uparrow \infty$, and

$$|p_m(x, f(x)) - p_m(y, f(y))| \leqq \Phi(\zeta, x, y) \tag{4}$$

for all $x, y \in Y \sim \{\zeta\}$ and all m.

Since T annihilates $\mathbb{C}[z, f]$, the dominated convergence theorem and this second property show that

$$\int_{Y \times Y} \{\Omega(\zeta, x) - \Omega(\zeta, y)\} \frac{d\mu(x,y)}{|x-y|^\alpha} = 0$$

for $\mathscr{L}^{2n}$ almost all $\zeta \in U$. (Note that μ has mass zero on $\{\zeta\} \times Y \cup Y \times \{\zeta\}$ for $\mathscr{L}^{2n}$ almost all ζ.) Hence $T\varphi = 0$ for all $\varphi \in \mathscr{D}(U)$, as required.

The forms $\Omega(\zeta, z)$ are Cauchy-Fantappié forms, construced in Weinstock's manner [14, 16]. It is merely necessary to demonstrate that this forms have the properties we need, which are more stringent than his requirements. To this end, we review his construction. Since f is continuously-differentiable at a and $f_{\bar{z}}(a)$ is invertible, there exists a neighbourhood U_1 of a such that the function

$$g(\zeta, z, w) = -\langle \zeta - z, f_{\bar{z}}(a)^{-1}\{f(\zeta) - w - f_z(a)(\zeta - z)\}\rangle$$

satisfies

$$\tilde{g}(\zeta, z) = -|z-\zeta|^2 + q(\zeta, z) \tag{5}$$

for all $\zeta, z \in U_1$, where

$$|q(\zeta, z)| \leqq M_2 |z-\zeta|^2, \tag{6}$$

where $M_2 < 1$ depends on a but not on ζ or z. Here $\tilde{g}(\zeta, z)$ stands for $g(\zeta, z, f(z))$; in general $\tilde{u}(z)$ shall denote $u(z, f(z))$, i.e. $\tilde{u}$ is the function on a neighbourhood of Y induced by a function u on a neighbourhood of X.

Next, by solving a Cousin problem with smooth dependence on a parameter, Weinstock finds a neighbourhood W_1 of X in $\mathbb{C}^{2n}$ and functions $H(\zeta, z, w)$ on $U_1 \times W_1$ and $\kappa(\zeta, z, w)$ on $U_1 \times U_1 \times U_1$ such that (a) both are C^1 in ζ and analytic in (z, w), (b) κ is nonvanishing, (c) $H = \kappa g$ on $U_1 \times U_1 \times U_1$, and (d) there exists $M_3 > 0$ such that $\tilde{H}(\zeta, z)$ takes no value in the sector

$$\{\omega \in \mathbb{C} : 0 < |\omega| < M_3, |\mathrm{Im}\,\omega| \leqq M_3 \,\mathrm{Re}\,\omega\}$$

for any $\zeta \in U_1$ and $z \in Y$.

By [15], there exist functions $H_k(\zeta, z, w)$, C^1 in $\zeta \in U_1$ and analytic in (z, w) on a neighbourhood $W_2 \subset W_1$ of X, such that

$$H(\zeta, z, w) = \sum_{k=1}^{n} (\zeta_k - z_k) H_k(\zeta, z, w).$$

He chooses a neighbourhood N of Y such that $(z, f(z)) \in W_2$ for all $z \in N$, and defines $G(\zeta, z) = \tilde{H}(\zeta, z)$, $G_k = \tilde{H}_k$, and the form $\Omega(\zeta, z)$ as

$$\frac{(n-1)!}{2\pi i} \sum_{k=1}^{n} (-1)^{k-1} G_k G^{-n} \bar{\partial}_\zeta G_1 \wedge \ldots \wedge \bar{\partial}_\zeta G_{k-1} \wedge \bar{\partial}_\zeta G_{k+1} \wedge \ldots \wedge \bar{\partial}_\zeta G_n \wedge d\zeta_1 \wedge \ldots \wedge d\zeta_n .$$

Let U be a bounded neighbourhood of a with $\operatorname{clos} U \subset U_1$.

By (5) and (6), $|G_k(\zeta, z)| \leqq M_4 |z-\zeta|$ on $U \times N$, where $M_4 > 0$ is independent of ζ and z. Also $|\kappa|$ is bounded below on $U \times U \times U$, hence

$$|G| \geqq M_5 |z-\zeta|^2 \tag{7}$$

for some $M_5 > 0$, independent of $(\zeta, z) \in U \times N$. Let

$$\bar{\partial}_\zeta H_1 \wedge \ldots \wedge \bar{\partial}_\zeta H_{k-1} \wedge \bar{\partial}_\zeta H_{k+1} \wedge \ldots \wedge \bar{\partial}_\zeta H_n = \sum_{j=1}^{n} R_{kj}(\zeta, z, w) d\bar{\zeta}_1 \wedge \ldots \wedge d\bar{\zeta}_{j-1} \wedge d\bar{\zeta}_{j+1} \wedge \ldots \wedge d\bar{\zeta}_n ,$$

$$S_j(\zeta, z, w) = \frac{(n-1)!}{2\pi i} \sum_{k=1}^{n} (-1)^{k-1} H_k R_{kj} .$$

Then S_j is continuous in $\zeta \in U_1$ and analytic in $(z, w) \in W_2$, and $K_j(\zeta, z) = \tilde{S}_j G^{-n}$.

To prove the first property of Ω, we need only show that for each $j \in \{1, \ldots, n\}$,

$$\int_U |\tilde{S}_j(\zeta, x) G^{-n}(\zeta, x) - \tilde{S}_j(\zeta, y) G^{-n}(\zeta, y)| d\mathscr{L}^{2n}(\zeta) \leqq M_6 |x-y|^\alpha , \tag{8}$$

where $M_6 > 0$ is independent of $x, y \in Y$.

The Frechet derivatives $D_z \tilde{S}_j(\zeta, z)$ and $D_z G_j(\zeta, z)$ are uniformly bounded on $U \times Y$. Thus we have constants M_7 and M_8 such that

$$\begin{aligned} |\tilde{S}_j(\zeta, x) - \tilde{S}_j(\zeta, y)| &\leqq M_7 |x-y| \\ |G_j(\zeta, z) - G_j(\zeta, y)| &\leqq M_8 |x-y| . \end{aligned} \tag{9}$$

Since $G_j(\zeta, \zeta) = 0$, it follows that $\tilde{S}_j(\zeta, \zeta) = 0$, hence

$$\begin{aligned} |\tilde{S}_j(\zeta, x)| &\leqq M_7 |\zeta - x| \\ |G_j(\zeta, x)| &\leqq M_8 |\zeta - x| . \end{aligned} \tag{10}$$

The latter inequality permits us to conclude that

$$|G(\zeta, x) - G(\zeta, y)| \leqq M_9 |x-y| \max\{|\zeta - x|, |\zeta - y|\} \tag{11}$$

for a certain constant $M_9 > 0$.

Given the estimates (7)–(11), the estimate (8) is proved by using the old Frostman two-ball trick, much as in [6]. Indeed, the left-hand side is bounded by

$$M_{10} |x-y| \log |x-y|^{-1} .$$

This fact is possible well-known. Certainly, for the Bochner-Martinelli kernel ($G_j = \overline{z_j - \zeta_j}$) it is classical (cf. [10, p. 1175; 13, p. 442]); the argument for general G_j

is not essentially different. In any case, we think it worthwhile to indicate the details, since we have not found them elsewhere. Incidentially, the delicate Lipschitz estimates which sometimes arise with kernels for solving the $\bar{\partial}$ problem, as in [1, 3, 4, 9, 10, 13], are not in question here, because we have no boundary integral to deal with.

Let $2r=|x-y|$, $z=\dfrac{x+y}{2}$,

$$B_x=\left\{\zeta\in U:|\zeta-x|\leqq\frac{7r}{4}\right\},$$

$$B_y=\left\{\zeta\in U:|\zeta-y|\leqq\frac{7r}{4}\right\},$$

$$T_{xy}=U\sim B_x\sim B_y.$$

Then

$$\int_{\zeta\varepsilon U}|K_j(\zeta,x)-K_j(\zeta,y)|d\mathscr{L}^{2n}(\zeta)\leqq\int_{B_x}+\int_{B_y}+\int_{T_{xy}}.$$

Now (suppressing ζ)

$$\begin{aligned}|K_j(x)-K_j(y)|&\leqq|\tilde{S}_j(x)|\,|G^{-n}(x)-G^{-n}(y)|+|\tilde{S}_j(x)-\tilde{S}_j(y)|\,|G^{-n}(y)|\\&\leqq M_7|\zeta-x|\sum_{k=0}^{n-1}|G(x)|^{-k-1}|G(y)|^{-n+k}|G(x)-G(y)|\\&\quad+M_7r|G(y)|^{-n}\end{aligned}$$

Thus for $\zeta\in B_x$,

$$|K_j(x)-K_j(y)|\leqq M_{11}\sum_{k=0}^{n-1}r^{-2n+2k+2}|\zeta-x|^{-2k-1}+M_{12}r^{-2n+1},$$

hence

$$\begin{aligned}\int_{B_x}|K_j(x)-K_j(y)|d\mathscr{L}^{2n}(\zeta)&\leqq M_{13}\sum_{k=0}^{n-1}r^{-2n+2k+2}\int_0^r s^{2n-2k-2}ds+M_{14}r\\&=M_{15}r.\end{aligned}$$

Similarly

$$\int_{B_y}|K_j(x)-K_j(y)|d\mathscr{L}^{2n}(\zeta)\leqq M_{15}r.$$

On T_{xy}, we have

$$M_{16}|\zeta-z|\leqq|\zeta-x|\leqq M_{17}|\zeta-z|,$$

$$M_{16}|\zeta-z|\leqq|\zeta-y|\leqq M_{17}|\zeta-z|,$$

$$\begin{aligned}|K_j(x)-K_j(y)|&\leqq M_{17}|G^{-n}(x)-G^{-n}(y)|+|\tilde{S}_j(x)-\tilde{S}_j(y)|\,|G^{-n}(y)|\\&\leqq M_{18}|\zeta-z|\sum_{k=0}^{n-1}|\zeta-z|^{-2n-2}\cdot|\zeta-z|\,|x-y|+M_{19}|x-y|\,|\zeta-z|^{-2n}\\&=M_{20}r|\zeta-z|^{-2n}.\end{aligned}$$

Thus if $d = \operatorname{diam} Y$, we have

$$\int_{T_{xy}} |K_j(x) - K_j(y)| d\mathscr{L}^{2n}(\zeta) \leqq \int_{\frac{3r}{4} \leqq |\zeta - z| \leqq d} M_{20} r |\zeta - z|^{-2n} d\mathscr{L}^{2n}(\zeta)$$

$$= M_{21} r \int_{\frac{3r}{4}}^{d} \frac{ds}{s}$$

$$\leqq M_{22}(1 + \log_+ r^{-1}) r .$$

The estimate (8) follows.

Now we turn to the second property of Ω. From the details of the above proof, we note that

$$\int_{\zeta \varepsilon U} |\tilde{S}_j(x)| \sum_{k=0}^{n-1} |G(x)|^{-k-1} |G(y)|^{-n+k} |G(x) - G(y)| d\mathscr{L}^{2n}(\zeta) \leqq M_{23} |x - y|^{\alpha}$$

for some $M_{23} > 0$ and all $x, y \in Y$.

Using (7), we see that the integral

$$\int_{\zeta \varepsilon U} |\zeta - x|^{1-\alpha} |G(\zeta, x)|^{-n} d\mathscr{L}^{2n}(\zeta)$$

is bounded independently of $x \in Y$. Let $\Phi_1(\zeta, x, y)$ denote

$$\sup_j |\tilde{S}_j(x)| \sum_{k=0}^{n-1} |G(x)|^{-k-1} |G(y)|^{-n+k} |G(x) - G(y)|$$
$$+ \sup_j |\tilde{S}_j(y)| \sum_{k=0}^{n-1} |G(y)|^{-k-1} |G(x)|^{-n+k} |G(x) - G(y)|$$
$$+ |x - y|^{\alpha} |\zeta - x|^{1-\alpha} |G(x)|^{-n} + |x - y|^{\alpha} |\zeta - y|^{1-\alpha} |G(y)|^{-n}$$
$$+ |x - y| .$$

Then

$$\int_{\zeta \varepsilon U} \Phi_1(\zeta, x, y) d\mathscr{L}^{2n}(\zeta) \leqq M_{24} |x - y|^{\alpha} ,$$

and hence, by Fubini's theorem,

$$\int_{Y \times Y} \Phi_1(\zeta, x, y) \frac{d|\mu|(x, y)}{|x - y|^{\alpha}} < +\infty \tag{12}$$

for $\mathscr{L}^{2n}$ almost all $\zeta \in U$.

Fix $\zeta \in U$ such that (12) holds. We proceed to establish the second property of Ω by using a suitable multiple of $\Phi_1(\zeta, x, y)$ for Φ.

Let $Q_m(z, w) = (H(\zeta, z, w) - m^{-1})^{-1}$, $m = 1, 2, 3, \ldots$. By the omitted sector property (d), there exist $m_0 > 0$ and $M_{25} > 0$ such that Q_m is analytic on a neighbourhood of X and

$$|\tilde{Q}_m(z)| \leqq M_{25} |G(\zeta, z)|^{-1}$$

whenever $m \geqq m_0$ and $z \in Y$. Also $\tilde{Q}_m(z) \to G^{-1}(\zeta, z)$ pointwise on $Y \sim \{\zeta\}$ as $m \uparrow \infty$.

Fix $j\in\{1,\ldots,n\}$. Fix x and y belonging to $Y\sim\{\zeta\}$. Then $|x-y|\leqq 2|x-\zeta|$ or $|x-y|\leqq 2|y-\zeta|$, for otherwise the triangle inequality yields $|x-y|<|x-y|$. Suppose $|x-y|\leqq 2|y-\zeta|$. Then for $m\geqq m_0$ we have

$$\begin{aligned}
&|\tilde{S}_j(\zeta,x)\tilde{Q}^n_m(x)-\tilde{S}_j(\zeta,y)\tilde{Q}^n_m(y)|\\
&\quad\leqq|\tilde{S}_j(\zeta,x)|\,|\tilde{Q}^n_m(x)-\tilde{Q}^n_m(y)|+|\tilde{S}_j(\zeta,x)-\tilde{S}_j(\zeta,y)|\,|\tilde{Q}^n_m(y)|\\
&\quad\leqq|\tilde{S}_j(\zeta,x)|\sum_{k=0}^{n-1}|\tilde{Q}^{k+1}_m(x)\tilde{Q}^{n-k}_m(y)|\,|G(x)-G(y)|+M_7M^n_{25}|x-y|\,|G(\zeta,y)|^{-n}\\
&\quad\leqq M^n_{25}|\tilde{S}_j(x)|\sum_{k=0}^{n-1}|G(x)|^{-k-1}|G(y)|^{k-n}|G(x)-G(y)|\\
&\qquad+M_7M^n_{25}2^{1-\alpha}|x-y|^\alpha|\zeta-y|^{1-\alpha}|G(\zeta,y)|^{-n}\\
&\quad\leqq M_{26}\Phi_1(\zeta,x,y).
\end{aligned}$$

Similarly, if $|x-y|\leqq 2|x-\zeta|$, we obtain the same estimate. Thus the estimate holds for all $x,y\in Y\sim\{\zeta\}$.

Now $S_j(\zeta,z,w)Q^n_m(z,w)$ is analytic on a neighbourhood of X. Since X is polynomially-convex, it is the joint spectrum in the Banach algebra $\mathrm{Lip}(1,Y)$ of the elements $z_1,\ldots,z_n,f_1,\ldots,f_n$. By the functional calculus for Banach algebras, there exists a polynomial $p_m(z,w)\in C[z,w]$ such that $\|\tilde{p}_m-\tilde{S}_j\tilde{Q}^n_m\|<m^{-1}$. Thus for $m>m_0$,

$$\begin{aligned}
|\tilde{p}_m(x)-\tilde{p}_m(y)|&\leqq 2\cdot M^{2n}_{11}\cdot\|\tilde{S}_j\|\cdot\Phi(x,y)+m^{-1}|x-y|\\
&\leqq\{2\cdot M^{2n}_{11}\|\tilde{S}_j\|+1\}\Phi(x,y).
\end{aligned}$$

Thus (4) holds. This proves the second property and concludes the proof of the theorem.

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