Restriction Algebras

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The preprint follows.
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RESTRICTIONS OF UNIFORM ALGEBRAS

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1. The diagram. Let A be a complex uniform algebra on a compact Hausdorff space X [2, p. 32]. This paper is about the family of sets $E \subset X$ for which the restriction algebra $A|E$ is uniformly closed. For background, cf. [1;2;3;5;7;8]. More details of the results described here will appear in the Proceedings of the Royal Irish Academy.

← In the end, fewer details appeared there!

The algebra A induces certain families of subsets of X, as follows:

$\mathcal{W}$ is the family of *weak-star* closed subsets of X, regarding X as a subset of the dual A^* (the "original" closed sets).

$\mathcal{C}$ is the family of A-*convex* subsets of X, i.e. $E \in \mathcal{C}$ if and only if E equals the A-*convex hull* of E (in X), which is the set $\{a \in X : |f(a)| \le \|f\|_E$ whenever $f \in A\}$, where $\|f\|_E$ denotes the uniform norm of f on E.

$\mathcal{H}$ is the family of *hull-kernel closed* subsets of X, i.e. $E \in \mathcal{H}$ if and only if $E = \{a \in X : f(a) = 0$ whenever $f \in \ker E\}$, where $\ker E = \{f \in A : f|E = 0\}$ ($f|E$ denotes the restriction of f to E).

$\mathcal{R}$ is the family of A-convex sets E such that $A|E$ is closed. The reason we restrict attention to A-convex sets is that for any $E \subset X$, with A-convex hull F, the restriction algebra $A|E$ is closed if and only if $A|F$ is closed.

$\mathcal{R}_1$ is the family of A-convex sets E such that the quotient $A/\ker E$ is isometric to the restriction algebra $A|E$, i.e. given $f \in A$ and $\varepsilon > 0$, there exists $g \in A$ such that $g|E = f|E$ and $\|g\|_X \le (1+\varepsilon)\|f\|_E$.

$\mathcal{J}$ is the family of *interpolation sets*, i.e. those $E \subset X$ such

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Abstract. Let A be a uniform algebra on a compact Hausdorff space X. We discuss the diagram of inclusions

$$\begin{array}{ccccccccccc} \mathcal{I}\cap\mathcal{P} & \to & \mathcal{I}_1 & \to & \mathcal{I} & & & & & & \\ \downarrow & & \downarrow & & \downarrow & & & & & & \\ \mathcal{P} & \to & \mathcal{R}_1 & \to & \mathcal{R} & \to & \mathcal{H} & \to & \mathcal{K} & \to & \mathcal{W} \end{array}$$

in which $\mathcal{W}$ is the family of weak-star closed subsets of X, $\mathcal{K}$ is the family of (relatively) A-convex sets, $\mathcal{H}$ is the family of (relatively) hull-kernel closed sets, $\mathcal{R}$ is the family of sets $E \in \mathcal{K}$ such that the restriction algebra $A|E$ is closed in C(E), $\mathcal{R}_1$ is the family of sets $E \in \mathcal{W}$ such that $A|E$ is isometric to A/ker E, $\mathcal{P}$ is the family of p-sets, $\mathcal{I}$ is the family of interpolation sets, i.e. sets $E \in \mathcal{W}$ such that $A|E = C(E)$, and $\mathcal{I}_1 = \mathcal{I}\cap\mathcal{R}_1$ is the family of isometric interpolation sets. We discuss the behaviour of these families under union and intersection. We give a new sufficient condition for the equality $\mathcal{H} = \mathcal{P}$. This yields a "Rudin-Carleson theorem" for hypodirichlet algebras of analytic functions.

§1. Introduction.

(1.1) Let A be a complex uniform algebra on a compact Hausdorff space X [2, p.32]. This paper is about the family of sets $E \subset X$ for which the restriction algebra $A|E$ is uniformly closed. For background, cf. [1,2,3,5,7,8].

The algebra A induces certain families of subsets of X, as follows:

(1) $\mathcal{W}$ is the family of weak-star closed subsets of X, regarding X as a subset of the dual A* (the "original" closed sets).

(2) $\mathcal{C}$ is the family of A-convex subsets of X, i.e. $E \in \mathcal{C}$ if and only if E equals the A-convex hull of E(in X), which is the set

$$\{a \in X : |f(a)| \leqslant \|f\|_E \text{ whenever } f \in A\},$$

where $\|f\|_E$ denotes the uniform norm of f on E.

(3) $\mathcal{H}$ is the family of hull-kernel closed subsets of X, i.e. $E \in \mathcal{H}$ if and only if

$$E = \{a \in X : f(a) = 0 \text{ whenever } f \in \ker E\},$$

where

$$\ker E = \{f \in A : f|E = 0\},$$

where $f|E$ denotes the restriction of f to E.

(4) $\mathcal{R}$ is the family of A-convex sets E such that $A|E$ is closed. The reason we restrict attention to A-convex sets is that for any

$E \subset X$, with A-convex hull F, the restriction algebra $A|E$ is closed if and only if $A|F$ is closed. This is proved in (2.2) below.

(5) $\mathcal{R}_1$ is the family of A-convex sets E such that the quotient A/ker E is isometric to the restriction algebra $A|E$, i.e. given $f \in A$ and $\varepsilon > 0$, there exists $g \in A$ such that $g|E = f|E$ and $\|g\|_X \leqslant (1+\varepsilon)\|f\|_E$.

(6) $\mathcal{I}$ is the family of <u>interpolation sets</u>, i.e. those $E \subset X$ such that E is weak-star closed and $A|E = C(E)$.

(7) $\mathcal{I}_1$ is the family of isometric interpolation sets, i.e. those $E \in \mathcal{I}$ such that C(E) is isometric to A/ker E.

(8) $\mathcal{P}$ is the family of p-<u>sets</u> in X, i.e. intersections of peak sets [2, p.56].

(9) $\mathcal{I} \cap \mathcal{P}$ is the family of p-<u>interpolation sets</u>.

The relationship between these families is expressed in the following diagram, in which each arrow is an inclusion map.

$$\begin{array}{ccccccccccc} \mathcal{I} \cap \mathcal{P} & \to & \mathcal{I}_1 & \to & \mathcal{I} & & & & & & \\ \downarrow & & \downarrow & & \downarrow & & & & & & \\ \mathcal{P} & \to & \mathcal{R}_1 & \to & \mathcal{R} & \to & \mathcal{H} & \to & \mathcal{C} & \to & \mathcal{W} \end{array}$$

(1.2) Some of these inclusions are obvious. The inclusion $\mathcal{P} \to \mathcal{R}_1$ is due to Glicksberg [5;2, p.58]. It implies the inclusion $\mathcal{I} \cap \mathcal{P} \to \mathcal{I}_1$.

The inclusion $\mathcal{R} \to \mathcal{H}$ is due to Bernard [1]. The inclusion $\mathcal{S} \to \mathcal{R}$ implies the inclusion $\mathcal{S}_1 \to \mathcal{R}_1$. We include brief proofs of the inclusions $\mathcal{R} \to \mathcal{H}$ and $\mathcal{S} \to \mathcal{R}$ in (2.4) and (2.5).

(1.3) Each inclusion in the diagram may be proper. We are interested mainly in the case when X = M(A), the maximal ideal space of A, and the case when X = Ш (A), the Šilov boundary of A. In both cases, each of the inclusions in the diagram may be proper. We give examples in (2.6) - (2.11).

(1.4) Each of the families $\mathcal{S} \cap \mathcal{P}$, $\mathcal{P}$, $\mathcal{H}$, $\mathcal{B}$, and $\mathcal{W}$ is known to be an F-topology, i.e. is closed under finite unions and arbitrary intersections. Clearly, the families $\mathcal{S}$ and $\mathcal{S}_1$ are directed downward (i.e. closed under subsets), and hence are closed under arbitrary intersections. In (3.1) and (3.2) we give examples to show that none of $\mathcal{R}$, $\mathcal{R}_1$, $\mathcal{S}$, $\mathcal{S}_1$ need be closed under finite unions, and that $\mathcal{R}$ and $\mathcal{R}$ may fail to be closed under finite intersections. In (3.3) we show that if X = M(A), E and F are <u>disjoint</u> elements of $\mathcal{R}$, then $E \cup F \in \mathcal{R}$.

We mention some other positive results. Glicksberg [5, pp.424-425] noted that $E \in \mathcal{P}$ and $F \in \mathcal{R}$ imply $E \cup F \in \mathcal{R}$. A similar argument shows that $E \cap F \in \mathcal{R}$ also. The family $\mathcal{R}_1$ is closed under nested intersection.

(1.5) Consider the problem, under what circumstances equality obtains between $\mathcal{R}$ or $\mathcal{S}$ and one of the other families in the diagram. Apart from the fact that a diagram of inclusions always prompts such a question, there is a natural reason for studying it: the families $\mathcal{P}$,

$\mathcal{H}$, $\mathcal{L}$, $\mathcal{W}$ may admit explicit description, so that an equality permits the explicit description of $\mathcal{R}$ or $\mathcal{S}$. Consider, for instance, the disc algebra A on the unit circle X. Then $\mathcal{P} = \mathcal{R}_1 = \mathcal{R} = \mathcal{H}$, and $\mathcal{S} \cap \mathcal{P} = \mathcal{S}_1 = \mathcal{S}$, and $\mathcal{R} = \mathcal{S} \cup \{X\}$. The family $\mathcal{S}$ consists of the closed sets of length zero. These facts are the Luzin-Privaloff and Rudin-Carleson theorems. They extend to the polydisc [7] and ball [8] in C^n. Glicksberg showed that, in general, $\mathcal{P}$ is the family of sets $E \in \mathcal{W}$ such that $\mu|E$ annihilates A whenever the measure μ annihilates A. This yields even more explicit descriptions of $\mathcal{P}$ in special situations. Thus interest focuses on the equation $\mathcal{R} = \mathcal{P}$. Glicksberg showed that $\mathcal{R} = \mathcal{P}$ if X = Ш(A) and A is logmodular, and Bernard obtained the stronger conclusion $\mathcal{H} = \mathcal{P}$ under the assumption that A has unique representing measures on X = Ш(A) and that M(A) is "bien-partagé" (the representing measure for each point of each Gleason part P of A is supported on the weak-star closure of P in M(A)). These results apply to many examples, but they fail to cover such simple algebras as A(U), where U is an annulus. In (4.1) we prove a result which yields a "Rudin-Carleson theorem" for A(U) whenever U is a plane domain bounded by a finite number of closed curves, and which also covers some infinitely-connected U.

§2. The diagram.

(2.1) *Lemma*. *Let* $E \subset X$ *and let* $A|E$ *be closed*. *Then* hull ker E *equals the* A-*convex hull of* E, *and* $A|F$ *is closed whenever* $E \subset F \subset$ hull ker E.

Proof. By the open mapping theorem, $A|E$ is closed if and only if there exists $M > 0$ such that each $f \in A|E$ has an extension $g \in A$ such that $\|g\|_X \leq M\|f\|_E$. Let $A|E$ be closed, let M be so chosen, and let $E \subset F \subset$ hull ker E. Let $f \in A$. Then $f|E$ has an extension $g \in A$ such that $\|g\|_X \leq M\|f\|_E$. Since $f = g$ on E, it follows that $f = g$ on F, so g extends $f|F$ and $\|g\|_X \leq M\|f\|_F$. Thus $A|F$ is closed.

In general, hull ker E contains the A-convex hull of E, since $\mathcal{H} \subset \mathcal{B}$. If $A|E$ is closed, if M is as above, and $F =$ hull ker E, then clearly $\|f\|_F \leq M\|f\|_E$ for all $f \in A$. Replacing f by f^n and taking roots and limits we conclude that $\|f\|_F \leq \|f\|_E$, so that F is contained in the A-convex hull of E. This completes the proof.

(2.2) *Theorem*. *Let* $E \subset X$, *and let* F *be the* A-*convex hull of* E. *Then* $A|E$ *is closed if and only if* $A|F$ *is closed*.

Proof. In view of the lemma, it remains to prove the "if" part

Suppose $A|F$ is closed. Choose $M > 0$ such that for each $f \in A$ the function $f|F$ has an extension $g \in A$ such that $\|g\|_X \leq M\|f\|_F$. Then, since $\|f\|_F \leq \|f\|_E$, we see that $A|E$ is closed.

(2.3) The above theorem fails if the A-convex hull is replaced by the hull-kernel closure. Consider, for example, the disc algebra A on either the unit circle $X = Ш(A)$ or the closed disc $X = M(A)$. If E is a closed semicircle, then $A|E$ is not closed, yet hull ker $E = X$.

(2.4) Corollary. $\mathcal{R} \subset \mathcal{H}$.

Proof. This is immediate from Lemma (2.1).

(2.5) Corollary. $\mathcal{J} \subset \mathcal{R}$.

Proof. Let $E \in \mathcal{J}$, and let F be the A-convex hull of E. Then $A|F$ is isometrically isomorphic to $A|E = C(E)$. If $F \sim E$ were nonempty, then $C(E)$ would admit an algebra homomorphism onto $\mathbb{C}$, other than evaluation at points of E, which is impossible.

(2.6) Example. $\mathcal{B} \neq \mathcal{W}$.

It is easy to see that $\mathcal{B}$ need not equal $\mathcal{W}$. It is less obvious that $\mathcal{B}$ need not equal $\mathcal{W}$ if $X = Ш(A)$. An example is the "thumbtack" [8, p.206].

Using Bishop's criterion [2, p.59] it is easy to see that $\mathcal{B} = \mathcal{W}$ if and only if each point of X is a p-point (generalized peak point).

(2.7) Example. $\mathcal{H} \neq \mathcal{B}$. The disc algebra on $Ш(A)$

(2.8) Example. $\mathcal{R} \neq \mathcal{H}$.

In (3.1) below we give an example in which there exist $E, F \in \mathcal{R}$ such that $E \cup F \notin \mathcal{R}$. Since $\mathcal{H}$ is an F-topology, we must have $\mathcal{R} \neq \mathcal{H}$.

(2.9) Example. $\mathcal{G}_1 \neq \mathcal{G}$, hence $\mathcal{R}_1 \neq \mathcal{R}$.

If a and b belong to the same Gleason part of A, and $E = \{a,b\}$, then $E \in \mathcal{G} \sim \mathcal{G}_1$. Thus $\mathcal{G} \sim \mathcal{G}_1 \neq \emptyset$ whenever X meets some Gleason part in two or more points. This can occur even for $X = \text{Ш}(A)$. Take, for instance $X \subset \mathbb{C}$ with no interior such that $R(X) \neq C(X)$ [2, p.25; p.54; p.146].

(2.10) Example. $\mathcal{P} \cap \mathcal{G} \neq \mathcal{G}_1$, hence $\mathcal{P} \neq \mathcal{R}_1$.

Take $E = \{a\}$, where a is a non p-point on X. An R(X), as above, gives an example with $X = \text{Ш}(A)$.

(2.11) Example. $\mathcal{P} \cap \mathcal{G} = \mathcal{G}_1 = \mathcal{G} \neq \mathcal{R} = \mathcal{R}_1 = \mathcal{P} \cap \mathcal{R}$.

The disc algebra.

§3. Unions and intersections.

(3.1) Example. None of $\mathcal{I}_1$, $\mathcal{I}$, $\mathcal{R}_1$, $\mathcal{R}$ is closed under union.

We shall give an example of $E,F \in \mathcal{I}_1$ with $E \cup F \notin \mathcal{R}$.

Let a_n be a sequence of positive numbers, decreasing to zero, and let $r_n > 0$ be such that the closed discs D_n with centres a_n and radii r_n are disjoint. Let $X = \{0\} \cup \bigcup_{n=1}^{\infty} D_n$, and let A be the algebra A(X) of all functions, continuous on X and analytic on int D_n for each n. For each n, pick $b_n \in$ int D_n, $b_n \neq a_n$, such that the Gleason distance $\rho(a_n,b_n)$ is less than 4^{-n}. Let $E = \{0,a_1,a_2,\dots\}$ and $F = \{0,b_1,b_2,\dots\}$. Then, obviously, E and F are isometric interpolation sets, and $A|E \cup F$ is dense in $C(E \cup F)$. Suppose $E \cup F \in \mathcal{R}$. Then $A|E \cup F = C(E \cup F)$. But consider the function $f \in C(E \cup F)$, given by $f(a_n) = 0$, $f(b_n) = 2^{-n}$, $f(0) = 0$. If $g \in A$ and $g|E \cup F = f$, then

$$2^{-n} = |g(a_n) - g(b_n)| \leqslant 4^{-n}\|g\|_X$$

for each n, which is impossible.

This A is generated by one element.

By replacing the discs D_n by Swiss cheeses, we obtain an example in which $X = \mathcal{W}(A)$.

(3.2) Example. Neither $\mathcal{R}_1$ nor $\mathcal{R}$ is closed under intersection.

Let E and F be disjoint copies of the set X constructed in (3.1), and let X be the space obtained by identifying the 0 of E with the 0 of F, the a_n of E with the a_n of F, and the b_n of E with the b_n of F, for each n. Let A be the algebra of functions $f \in C(X)$ such that $f|E \in A(E)$ and $f|F \in A(F)$. Then E and F belong to $\mathcal{R}_1$, because $A|E = A(E)$ and $A|F = A(F)$, as is easily seen. But $A|E \cap F$ is $A(E)|E \cap F$, and is not closed, as we saw in (3.1). Thus $E \cap F \notin \mathcal{R}$.

(3.3) Theorem. *Let* $X = M(A)$ *and let* $E,F \in \mathcal{R}$, *with* $E \cap F = \emptyset$. *Then* $E \cup F \in \mathcal{R}$.

Proof. We employ the following characterisation of $\mathcal{R}$, due to Glicksberg [5]. Let $E \in \mathcal{L}$. Then $A|E$ is closed if and only if there exists $\kappa = \kappa(E) > 0$ such that

$$\text{dist}\left[\mu,(A|E)^{\perp}\right] \leqslant \kappa \,\text{dist}\left[\mu,A^{\perp}\right]$$

whenever μ is a measure supported on E, where $A^{\perp}$ denotes the space of measures on X which annihilate A, and $(A|E)^{\perp}$ denotes the space of measures on E which annihilate A.

Let $\kappa(E)$ and $\kappa(F)$ be chosen as above.

Since E and F are disjoint and hull-kernel closed, the ideal ker E + ker F is contained in no maximal ideal. Since A is a Banach algebra,

$$A = \ker E + \ker F.$$

Choose $f \in \ker E$ and $g \in \ker F$ such that $f+g = 1$.

Let μ be a measure supported on $E \cup F$. Then $\mu = f\mu + g\mu$, $f\mu$ is supported on F and $g\mu$ is supported on E. There exist annihilating measures σ and τ, supported on E and F, respectively, such that

$$\|\sigma-g\mu\| \leqslant \kappa(E)\,\text{dist}\left[g\mu,A^{\perp}\right],$$

$$\|\tau-f\mu\| \leqslant \kappa(F)\,\text{dist}\left[f\mu,A^{\perp}\right].$$

Now clearly $\operatorname{dist}\left[f\mu, A^{\perp}\right] \leqslant \|f\|_X \operatorname{dist}\left[\mu, A^{\perp}\right]$, since $f\lambda \in A^{\perp}$ whenever $\lambda \in A^{\perp}$, and similarly $\operatorname{dist}\left[g\mu, A^{\perp}\right] \leqslant \|g\|_X \operatorname{dist}\left[\mu, A^{\perp}\right]$. Thus

$$\operatorname{dist}\left[\mu, (A|E \cup F)^{\perp}\right]$$

$$\leqslant \|\mu - \sigma - \tau\|$$

$$\leqslant \left\{\kappa(E)\|g\|_X + \kappa(F)\|f\|_X\right\} \operatorname{dist}\left[\mu, A^{\perp}\right].$$

Thus, by Glicksberg's characterisation, $E \cup F \in \mathcal{R}$.

§4. Equality between different classes.

(4.1) *Theorem*. *Let* A *be a uniform algebra on* $X \subset M(A)$. *Suppose* (1) A *has no completely-singular annihilating measures on* X, (2) *for each* $a \in M(A)$, *each representing measure for* a *on* X *is supported on the hull-kernel closure in* M(A) *of the Gleason part of* a, *and* (3) *for each* $a \in M(A)$ *and each representing measure* ν *for* a *on* X, *there exists a Jensen measure* μ *for* a *on* X, *with* $\nu << \mu$. *Then* $\mathcal{H} = \mathcal{P}$.

Proof. Glicksberg [5;2, p.58] showed that $E \in \mathcal{P}$ if and only if $\mu|E \in A^{\perp}$ whenever $\mu \in A^{\perp}$. Bernard [1, p.377] deduced that if $E \in \mathcal{P}$, then E is *ergodic* for all representing measures for A on X, in the sense that if μ is such a measure, then $\mu(E)$ is 0 or 1. If A has no completely-singular annihilating measures, then by the general F. and M. Riesz theorem [2, p.45, (7.11)], if E is ergodic for all representing measures, then $E \in \mathcal{P}$. Thus in the present situation, $\mathcal{P}$ is the family of sets which are ergodic for all representing measures.

Suppose $E \in \mathcal{H}$. We wish to show that $E \in \mathcal{P}$. It suffices to take $E = f^{-1}(0)$ for some $f \in A$. Let τ be a representing measure for a point $a \in M(A)$. We wish to show that $\tau(E) = 0$ or 1. Suppose $\tau(E) > 0$. Let $P \subset M(A)$ be the Gleason part of a, and let b be any point of P. Then b has a representing measure ν on X such that $\tau \ll \nu$ [2, p.143,(1.2)], hence $\nu(E) > 0$. By hypothesis (3), there is a

Jensen measure μ for b with $\nu \ll \mu$, hence $\mu(E) > 0$, so that

$$\log|\hat{f}(b)| \leqslant \int \log|f| d\mu = -\infty ,$$

where $\hat{f}$ denotes the Gelfand transform of f. Thus $\hat{f}(b) = 0$. Thus $\hat{f} = 0$ on P, hence $\hat{f} = 0$ on the hull-kernel closure of P in M(A). By hypothesis (2), $\hat{f} = f = 0$ on the support of τ, hence $\tau(E) = 1$. The result follows.

(4.2) This result does not imply Glicksberg's result that $\mathcal{R} = \mathcal{P}$ for logmodular algebras. The hypothesis on the support of representing measures fails for any algebra with a one-point part off the Shilov boundary. It would be interesting to know if $\mathcal{R} = \mathcal{P}$ whenever A is hypodirichlet on $X = Ш(A)$ (cf. (4.4) below).

(4.3) Let U be an open subset of the Riemann sphere, and let A(U) denote the algebra of all functions continuous on clos U and analytic on U. Suppose A(U) has nonconstant functions in it. The maximal ideal space of A(U) is clos U (Arens' theorem). The Shilov boundary of A(U) is the <u>essential boundary</u> of U, i.e. the set of points a ε bdy U such that a is an essential singularity for some f ε A(U). The algebra A(U) has no completely singular annihilating measures, and representing measures for a point of clos U are always supported on the closure of the part [4]. Thus we obtain the following.

<u>Corollary</u>. <u>Let</u> A = A(U) <u>on</u> $X = Ш(A)$, <u>and suppose that each representing measure for each point</u> a ε clos U <u>is absolutely-continuous with respect to some Jensen measure for</u> a. <u>Then</u> $\mathcal{H} = \mathcal{R} = \mathcal{R}_1 = \mathcal{P}$.

(4.4) Corollary. Suppose $A = A(U)$ is a hypodirichlet algebra on $X = \text{bdy } U$. Then $\mathcal{H} = \mathcal{R} = \mathcal{R}_1 = \mathcal{P}$ is the family of all closed sets $E \subset X$ which are ergodic for harmonic measure for each component of U.

Proof. If A is a hypodirichlet on X, then $X = \mathcal{W}(A)$ and all representing measures on X for a point $a \in M(A)$ are dominated by the (unique) Jensen measure for a [2, p.116]. In case $A = A(U)$, the last corollary yields $\mathcal{H} = \mathcal{P}$. As we noted in the proof of (4.1), a closed set $E \subset X$ belongs to $\mathcal{P}$ if and only if E is ergodic for all representing measures. Thus $E \in \mathcal{P}$ if and only if E is ergodic for all Jensen measures. Since $\log|f|$ is subharmonic on U for each $f \in A(U)$, it follows that the Jensen measure for a point $a \in U$ is in fact the harmonic measure for a on the boundary of the component of a in U. Thus $E \in \mathcal{P}$ if and only if E is ergodic for all these harmonic measures. This completes the proof.

(4.5) Gamelin and Garnett [4] gave fairly explicit necessary and sufficient conditions on U for A(U) to be hypodirichlet.

The fact that $\mathcal{R} = \mathcal{P}$ for hypodirichlet A(U) does not follow from Glicksberg's result on logmodular algebras. Indeed A(U) is logmodular if and only if it is dirichlet.

Corollary (4.3) applies to some non-hypodirichlet A(U), such as the "champagne bubble" algebra [2, p.227]. This infinitely-connected U is such that harmonic measure dominates all representing measures, and is, as always, a Jensen measure.

(4.6) Generalised Rudin-Carleson Theorem. Let A(U) be hypodirichlet on X = bdy U. Then $\mathcal{J} = \mathcal{J}_1 = \mathcal{J} \cap \mathcal{P}$ is the family of closed sets $E \subset X$ having harmonic measure zero with respect to each component of U.

Proof. Since $\mathcal{R} = \mathcal{P}$, it follows that $\mathcal{J} = \mathcal{J} \cap \mathcal{P}$.

Suppose $E \in \mathcal{J}$. By the last corollary, E is ergodic for all harmonic measures. If there exists a connected component V of U for which E has full harmonic measure, then E contains bdy V, hence $C(\text{bdy } V) = A|\text{bdy } V \subset A(V)|\text{bdy } V \neq C(\text{bdy } V)$, which is impossible. Thus E has harmonic measure zero for all components of U.

Conversely, if E has harmonic measure zero for all components, then $\mu|E = 0$ whenever $\mu \perp A$, by the generalized F. and M. Riesz theorem, hence $E \in \mathcal{J} \cap \mathcal{P}$ by [2, p.58].

(4.7) Problem. Glicksberg showed [6] that if A|E is closed for each weak-star closed set $E \subset X$, and X = M(A), then A = C(X). In view of Theorem (2.2), this shows that if $\mathcal{R} = \mathcal{L}$ and X = M(A), then A = C(X). This is one of a body of results which characterise C(X) among its subalgebras in various ways. We may ask whether $\mathcal{R} = \mathcal{H}$ and X = M(A) force A = C(X), i.e. whether "weak-star closed" may be replaced by "hull-kernel closed" in Glicksberg's result.

Smyth and West [9] asked a related question. They asked (essentially) whether a uniform algebra A on X = M(A), of which all quotients A/I by closed ideals are also uniform algebras, must be C(X). If all quotients are uniform algebras, then $\mathcal{R}_1 = \mathcal{H}$, so a positive answer to the first question implies a positive answer to the second.