Math. Ann. 246, 225–232 (1980)

Mathematische Annalen

Approximation by Polynomials in Two Complex Variables

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1. Introduction

(1.1) This paper is about approximation by analytic polynomials in two complex variables on certain compact sets in $\mathbb{C}^2$. Specifically, we consider graphs X of the form

$$X = \{(z, f(z)) \in \mathbb{C}^2 : z \in D\},$$

where $f(z)$ is a continuous complex-valued function defined on the closed unit disc D.

There are substantial results in this area, notably by Mergelyan [5], Wermer [14–16, Chap. 17], Hörmander and Wermer [4], Nirenberg and Wells [6, 7], Freeman [2], Preskenis [10–12], and Range and Siu [13]. The main open problem is the following: *Are the polynomials uniformly dense in $C(X)$ whenever f is a direction-reversing homeomorphism?* For smooth homeomorphisms f, the local formulation of "direction-reversing" is

$$|f_{\bar{z}}| > |f_z| \quad \text{on} \quad D,$$

where

$$f_{\bar{z}} = \tfrac{1}{2}(f_x + i f_y),$$
$$f_z = \tfrac{1}{2}(f_x - i f_y).$$

The weaker condition $f_{\bar{z}} \neq 0$ on D guarantees that the smooth surface X has no *complex tangents* (that is that the tangent plane at a point of X is never a complex line). Wermer [15] proved the following.

(1.2) Theorem. *Suppose f is C^1 on a neighbourhood of D, suppose X is polynomially-convex, and $f_{\bar{z}} \neq 0$ a.e. on D. Then the analytic polynomials are uniformly dense in $C(X)$.*

This sparked off two lines of investigation. One line is the search for conditions for polynomial convexity of X. The other line begins with the observation that the degree of smoothness of f in Wermer's result (one) is higher than the degree of

0025-5831/80/0246/0225/$01.60

approximation (zero, that is C^0, or uniform). If we assume that $f \in C^k$ and $f_{\bar{z}} \neq 0$ everywhere, then what degree of approximation can be achieved? Assuming X to be polynomially-convex, the functional calculus [3, (III.4)] shows that we need only consider approximation by elements of $\mathcal{O}(X)$, the algebra of all functions holomorphic on a neighbourhood of X. The culmination of this line is the following result of Range and Siu [13].

(1.3) Theorem. *Let $f \in C^k(D)$ and $f_{\bar{z}} \neq 0$ everywhere on D. Then $\mathcal{O}(X)$ is dense in $C^k(X)$, in C^k norm.*

The hypothesis on $f_{\bar{z}}$ cannot be weakened. If $f_{\bar{z}}$ vanishes at even one point $a \in D$, then all functions in $\mathcal{O}(X)$ satisfy the tangential Cauchy-Riemann equations at the point $(a, f(a))$, and all C^1 limits of such functions will inherit the same property.

What is the best result obtainable with the hypotheses of Wermer's theorem (1.2)? We cannot expect C^1 approximation. Can we improve on uniform approximation? We shall prove the following result.

(1.4) Theorem. *Suppose $f \in \mathrm{Lip}(1, D)$, suppose $f_{\bar{z}} \neq 0$ a.e. in D, and suppose*

$$X = \{(z, f(z)) : z \in D\}$$

is polynomially-convex. Then for all α with $0 < \alpha < 1$, the polynomials are dense in $\mathrm{lip}(\alpha, X)$.

The Lipschitz spaces $\mathrm{Lip}(1, D)$ and $\mathrm{lip}(\alpha, X)$ are defined as follows. Let (E, ϱ) be a metric space, and let $0 < \alpha \leqq 1$. Then $\mathrm{Lip}(\alpha, E)$ is the space of bounded functions $g : E \to \mathbb{C}$ such that for some $\kappa > 0$,

$$|g(x) - g(y)| \leqq \kappa \varrho(x, y)^\alpha$$

for all $x, y \in E$, with the norm $\sup|g| +$ least κ. The space $\mathrm{lip}(\alpha, E)$ consists of those functions $g \in \mathrm{Lip}(\alpha, E)$ such that, given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|g(x) - g(y)| \leqq \varepsilon \varrho(x, y)^\alpha$$

whenever x and y satisfy $\varrho(x, y) < \delta$. If E is a subspace of a Euclidean space and $0 < \alpha < 1$, then $\mathrm{lip}(\alpha, E)$ is the closure in $\mathrm{Lip}(\alpha, E)$ of the C^∞ functions (restricted to E). Convergence in the $\mathrm{Lip}\,\alpha$ norm implies convergence in the uniform norm. Thus the conclusion of the theorem improves on that of (1.2).

We have relaxed the hypothesis on f from $f \in C^1(D)$ to $f \in \mathrm{Lip}(1, D)$. This is natural since $\mathrm{Lip}(1, D)$ functions are differentiable a.e., with bounded partials (Rademacher's theorem [1, p. 216]). There is no point in insisting that $f_{\bar{z}}$ exist *everywhere*, since we only place a restriction ($f_{\bar{z}} \neq 0$) on it *almost* everywhere.

The conclusion cannot be pushed to $\alpha = 1$. If X is smooth, then the closure of the C^∞ functions in $\mathrm{Lip}(1, X)$ is $C^1(X)$, and the Lip 1 norm is comparable to the C^1 norm on this closure. If $f_{\bar{z}} = 0$ at one point, then there are C^∞ functions which are not Lip 1 limits of polynomials.

Thus the degree of approximation almost, but not quite, matches the degree of smoothness of f.

In Sect. 2 we prove Theorem (1.4). The new tool we employ is the Cauchy transform on the dual of $\mathrm{Lip}\,\alpha$, which was introduced in [9]. Using this, we get the central Lemma (2.2), which shows that several, a priori different, degrees of approximation are actually equivalent. Given this lemma, the theorem is proved using techniques of Wermer.

In Sect. 3 we discuss polynomial convexity of X, and explicit approximation theorems.

2. Proof of Main Theorem

(2.1) Let $\mathbb{C}[z,w]$ denote the vectorspace of all polynomials in z and w, with complex coefficients. If $f\in \mathrm{Lip}(1,D)$, then the map $z\mapsto(z,f(z))$ is a bi-Lipschitzian map of D onto the graph X of f. This induces bi-continuous algebra homomorphisms of $C(X)$ onto $C(D)$ and of $\mathrm{Lip}(\alpha,X)$ onto $\mathrm{Lip}(\alpha,D)$ for $0<\alpha\leqq 1$. The map $\mathrm{Lip}(\alpha,X)\to\mathrm{Lip}(\alpha,D)$ maps $\mathrm{lip}(\alpha,X)$ onto $\mathrm{lip}(\alpha,D)$. Thus we obtain the following lemma.

Lemma. (i) *$\mathbb{C}[z,w]$ is uniformly dense in $C(X)$ if and only if $\mathbb{C}[z,f]$ is uniformly dense in $C(D)$.*

(ii) *$\mathbb{C}[z,w]$ is dense in $\mathrm{lip}(\alpha,X)$ if and only if $\mathbb{C}[z,f]$ is dense in $\mathrm{lip}(\alpha,D)$.*

(2.2) Let $\mathscr{L}^2$ denote Lebesgue measure on the plane. For $1\leqq p\leqq\infty$, let $L^p(D)$ denote the space $L^p(D,\mathscr{L}^2)$.

Lemma. *Let $f\in\mathrm{Lip}(1,D)$, and suppose $f_{\bar z}\neq 0$ a.e. in D. Then the following six conditions are equivalent.*

(1) *$\mathbb{C}[z,f]$ is dense in $\mathrm{lip}(\alpha,D)$ for all α with $0<\alpha<1$.*
(2) *There exists α such that $0<\alpha<1$ and $\mathbb{C}[z,f]$ is dense in $\mathrm{lip}(\alpha,D)$.*
(3) *$\mathbb{C}[z,f]$ is uniformly dense in $C(D)$.*
(4) *$\mathbb{C}[z,f]$ is weak-star dense in $L^\infty(D)$.*
(5) *$\mathbb{C}[z,f]$ is dense in $L^p(D)$ for all p with $2<p<\infty$.*
(6) *There exists p such that $2<p<\infty$ and $\mathbb{C}[z,f]$ is dense in $L^p(D)$.*

Proof. Obviously (1)⇒(2). Lip α convergence implies uniform convergence. Hence (2)⇒(3).

Uniform convergence implies weak-star convergence, and $C(D)$ is weak-star dense in $L^\infty(D)$, hence (3)⇒(4).

Suppose (4) holds. Let $q=\frac{p}{p-1}$. Then the dual of $L^p(D)$ is $L^q(D)$. Since $\mathscr{L}^2(D)<\infty$, we have $L^q(D)\subset L^1(D)$. By (4), there are no nonzero annihilators of $\mathbb{C}[z,f]$ in $L^1(D)$, and hence, a fortiori, none in $L^q(D)$. By the separation theorem, (5) holds. Thus (4)⇒(5).

Obviously (5)⇒(6).

To complete the proof, it suffices to show that (6)⇒(2) and (5)⇒(1). Both these assertions follow from the next lemma.

(2.3) Lemma. *Let $2<p<\infty$, and let*

$$0<\alpha<\frac{p-2}{p}.$$

Let $f \in \mathrm{Lip}(1, D)$, *and suppose* $f_{\bar{z}} \neq 0$ *a.e. in D. Suppose* $\mathbb{C}[z, f]$ *is dense in* $L^p(D)$. *Then* $\mathbb{C}[z, f]$ *is dense in* $\mathrm{lip}(\alpha, D)$.

Proof. Let $T \in \mathrm{Lip}(\alpha, D)^*$ be an annihilator of $\mathbb{C}[z, f]$. Consider the distribution $\hat{T}$, the Cauchy transform [9, §3] of T. Since T annihilates $\mathbb{C}[z]$, it follows that $\hat{T}$ is supported on D. By [9, p. 388], $\hat{T}$ may be represented by integration against a function in $L^q(D)$, where

$$q = \frac{1}{p-1},$$

since

$$1 < q < \frac{2}{1+\alpha}.$$

[Note the misprint on p. 388, line 16. Instead of $2(1+\alpha)$ it should read $2/(1+\alpha)$, as on line 12.]

As in [9], we denote this L^q function by $\hat{T}(z)$. For $g \in C^\infty$ we have

$$Tg = \int_D g_{\bar{z}} \hat{T} \, d\mathscr{L}^2. \tag{1}$$

Fix $g \in \mathrm{Lip}(1, D)$. Then $g \in \mathrm{lip}(\alpha, D)$. The space C^∞ is norm-dense in $\mathrm{lip}(\alpha, D)$, and weak-star dense in $\mathrm{Lip}(1, D)$, hence we can choose a sequence $g_n \in C^\infty$ such that $g_n \to g$ in the norm of $\mathrm{Lip}(\alpha, D)$ and $g_n \to g$ weak-star in $\mathrm{Lip}(1, D)$. Hence $g_{n_{\bar{z}}} \to g_{\bar{z}}$ weak-star in $L^\infty(D)$. Thus

$$\begin{aligned} Tg &= \lim_n Tg_n \\ &= \lim_n \int_D g_{n_{\bar{z}}} \hat{T} \, d\mathscr{L}^2 \\ &= \int_D g_{\bar{z}} \hat{T} \, d\mathscr{L}^2, \end{aligned}$$

since $\hat{T} \in L^q(D) \subset L^1(D)$. In other words, (1) holds for all $g \in \mathrm{Lip}(1, D)$.

Thus, for all nonnegative integers r and s, we have

$$\begin{aligned} 0 &= T(z^r f^{s+1}) \\ &= (s+1) \int_D z^r f^s f_{\bar{z}} T \, d\mathscr{L}^2, \end{aligned}$$

hence the function $f_{\bar{z}} \hat{T} \in L^q(D)$ annihilates $\mathbb{C}[z, f]$. Since $L^p(D)^* = L^q(D)$, this forces $f_{\bar{z}} \hat{T} = 0$ a.e. in D, hence $\hat{T} = 0$ a.e. in D, hence $\hat{T} = 0$ as a distribution. This implies that T annihilates $\mathrm{lip}(\alpha, D)$ [9, p. 376].

By the separation theorem, $\mathbb{C}[z, f]$ is dense in $\mathrm{lip}(\alpha, D)$.

(2.4) *Proof of* (*1.4*). By Lemma (2.2), it suffices to prove that $\mathbb{C}[z, f]$ is uniformly dense in $C(D)$. Wermer [15] proved this under the stronger assumptions $f \in C^1(D)$, and $f_{\bar{z}} \neq 0$ everywhere in D. He himself remarked (p. 8) that his proof only requires $f_{\bar{z}} \neq 0$ a.e. But it is also true that his proof works for $f \in \mathrm{Lip}(1, D)$, instead of $C^1(D)$. Indeed, his argument establishes the following fact, under the hypotheses of (1.4) (cf. [15 Lemma 4, p. 9]):

Let μ be an annihilating measure on D for $\mathbb{C}[z,f]$.

If (1) f is differentiable at a, (2) $f_{\bar z}(a) \neq 0$, and (3) $\int \frac{d|\mu|}{|z-a|} < \infty$, then the Cauchy transform $\hat\mu$ vanishes at a.

Since conditions (1)–(3) hold almost everywhere in D, it follows that $\hat\mu = 0$ a.e. in D whenever μ annihilates $\mathbb{C}[z,f]$. Since $\hat\mu = 0$ off D for any such μ, it follows that the only annihilating measure is the zero measure. The result follows.

3. Polynomial Convexity

(3.1) Fix $f \in \mathrm{Lip}(1, D)$, and let $X = \{(z, f(z)) : z \in D\}$. There are several known sets of sufficient conditions for X to be polynomially-convex. The main ones are as follows.

(1) (Mergelyan [10, Theorem (2.1)]) *f is real-valued, and each contour $f^{-1}(f(a))$ has no interior and has connected complement.*

(2) (Wermer [14]) *$f = \bar z + R$, where $R \in C(D)$ and*

$$|R(a) - R(b)| < |a - b|$$

for all $a \neq b$ in D.

(3) (Preskenis [10, (1.9)]) *$f \in C^1(D)$, $f_{\bar z} \neq 0$ everywhere in D, and for all $a \in D$ there exists ϕ_a, belonging to the uniform closure of $\mathbb{C}[z,f]$ on D, such that $\mathscr{L}^2 \phi_a^{-1}(0) = 0$ and*

$$\mathrm{Re}(z-a)\,\phi_a(z) \geqq 0$$

on D.

(4) (Preskenis [12]) *$\mathrm{Re} f_{\bar z} \geqq |f_z|$ almost everywhere, and each contour $f^{-1}(f(a))$ is countable.*

(5) (Preskenis [12]) *$f = \bar z^k \phi(|z|^{2k})$, where $\phi \in C^1[0, 1]$, k is a positive integer, and $|f_{\bar z}| > |f_z|$ on $D \sim \{0\}$.*

Adding (if necessary) the condition $f_{\bar z} \neq 0$ almost everywhere to any of these sets of conditions, we get an explicit approximation theorem.

In (3.2), we show that the second condition in (4) can be dropped.

In (3.4), we give a new condition for polynomial convexity, related to (5).

(3.2) Theorem. *Let $f \in \mathrm{Lip}(1, D)$, and suppose $\mathrm{Re} f_{\bar z} \geqq |f_z|$ a.e. in D. Then $X = \{(z, f(z)) : z \in D\}$ is polynomially-convex.*

Proof. We claim that

(1) $\mathrm{Re}\,(z-a)(f(z) - f(a)) \geqq 0$ for all $z, a \in D$.

Fix a. It suffices, by continuity, to prove (1) for a dense set of $z \in D$. For almost all $z \in D$, we have

(2) $\mathrm{Re} f_{\bar z} \geqq |f_z|$ almost everywhere (with respect to length) on the straight line from a to z. Let $z \in D$ be such that (2) holds. The straight line γ from a to z is

parametrised by

$$\zeta = a + (z-a)t, \qquad 0 \leqq t \leqq 1.$$

$$\begin{aligned}(z-a)&(f(z)-f(a))\\ &= (z-a)\int_\gamma df\\ &= (z-a)\int_0^1 \{f_z(\zeta)(z-a) + f_{\bar z}(\zeta)(\bar z - \bar a)\}\,dt\\ &= \int_0^1 \{|z-a|^2 f_{\bar z}(\zeta) + (z-a)^2 f_z(\zeta)\}\,dt.\end{aligned}$$

The integrand has non-negative real part for almost all t, hence (1) holds. The claim follows.

Next, suppose $(a,b)\in\mathbb{C}^2$ belongs to the polynomial hull of X. Then

$$|p(a,b)| \leqq \sup_X |p|$$

for every polynomial $p(z,w)$. Taking $p(z,w)=z$, we see that $|a|\leqq 1$. Thus hull X is contained in $X\times\mathbb{C}$.

There are two cases to consider.

Case 1. $|a|=1$.

In this case, the polynomial $1/2(\bar a z+1)$ (restricted to hull$X$) peaks on $E=\{a\}\times\mathbb{C}$. This forces

$$\begin{aligned}E\cap \operatorname{hull} X &= \operatorname{hull}(E\cap X)\\ &= \{(a, f(a))\}.\end{aligned}$$

Thus $b=f(a)$, and $(a,b)\in X$.

Case 2. $|a|<1$.

Suppose $b\neq f(a)$.

The function $(z-a)(f(z)-f(a))$ maps X to the right half-plane. In view of the equation

$$(z-a)(f(z)-b) = (z-a)(f(z)-f(a)) + (z-a)(f(a)-b),$$

we see that the left-hand side takes some values in the left half-plane, for $z\in D$, because the second term dominates the first for z near a. Choose $c\in D$ such that

$$\operatorname{Re}(c-a)(f(c)-b)<0,$$

and define

$$h(z,w) = (c-z)(f(c)-w).$$

Then

$$\operatorname{Re} h(a,b) < 0 \leqq \operatorname{Re} h(z,w)$$

for all $(z,w)\in X$, hence

$$\sup_X |\exp\{-h\}| < |\exp\{-h(a,b)\}|,$$

hence $(a,b)\notin \mathrm{hull}X$. This contradiction shows that $b=f(a)$, so that $(a,b)\in X$.

Thus, in each case, $(a,b)\in X$. Hence hull $X=X$, and we are done.

(3.3) Corollary. *Let* $f\in \mathrm{Lip}(1,D)$, *and suppose* $f_{\bar z}\neq 0$ *a.e. and* $\mathrm{Re}\, f_{\bar z}\geqq |f_z|$ *a.e. Then* $\mathbb{C}[z,w]$ *is dense in* $\mathrm{lip}(\alpha,X)$.

Proof. Combine (1.4) and (3.2).

We remark that the hypotheses could also be stated: $\mathrm{Re}\, f_{\bar z}\geqq |f_z|$ a.e. and the set of critical points of f has area zero.

(3.4) Theorem. *Let* g *be continuous on* D *and analytic on the interior of* D. *Let* $[a,b]=|g|^2(D)$. *Let* $\phi:[a,b]\to\mathbb{C}$ *be continuous. Let* $f=\bar g\phi(|g|^2)$. *Suppose* $t\phi(t)$ *is one-to-one. Then* $\mathbb{C}[z,f]$ *is dense in* $C(D)$, *and hence* X *is polynomially-convex.*

Proof. The algebra $\mathbb{C}[z,f]$ contains $|g|^2\phi(|g|^2)$. The function $t\phi(t)$ maps $[0,1]$ homeomorphically onto an arc Γ. By Lavrentiev's theorem [10, (2.2)], the polynomials are uniformly dense in $C(\Gamma)$. Thus $|g|$ is a uniform limit on D of polynomials in $|g|^2\phi(|g|^2)$. Hence clos $\mathbb{C}[z,f]$ is a $\mathbb{C}[|g|]$-module. By [8, p. 234], it suffices to prove that $\mathbb{C}[z,f]$ is dense in $C(E)$, for each contour $E=\{|g|=\alpha\}$ of $|g|$.

Fix $\alpha\in \mathrm{im}\,|g|$.

If $\phi(\alpha^2)\neq 0$, then on E

$$\mathbb{C}[z,f]=\mathbb{C}[z,\bar g\phi(\alpha^2)]=\mathbb{C}[z,\bar g],$$

and this is dense in $C(E)$ by [10, Corollary (2.5)].

If $\phi(\alpha^2)=0$, then $\mathbb{C}[z,f]=\mathbb{C}[z]$ on E. Since $0\phi(0)=\alpha\phi(\alpha^2)$, we have $\alpha=0$, hence $E=g^{-1}(0)$. Thus E is the union of a discrete subset of int D and a closed subset of the unit circle having length zero. Hence, E has no interior and connected complement, and by Lavrentiev's theorem, $\mathbb{C}[z]$ is uniformly dense in $C(E)$.

The proof is complete.

(3.5) Combining the method of (3.4) with results from [8], we can prove the following related result.

Theorem. *If the hypotheses of Theorem (3.4) hold, and* g *is conformal on* D, *then the vectorspace sum*

$$\mathbb{C}[z,|g|^2]+\mathbb{C}[|g|^2,f]$$

is uniformly dense in $C(D)$.

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Received June 5, 1979