16.

REAL PARTS OF ANALYTIC FUNCTIONS

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Abstract

We give a sufficient condition on a bijection $\phi : \mathbb{R} \to \mathbb{R}$ so that

$$\mathrm{Re}\, H^\infty = \mathrm{Re}\, H^\infty \circ \phi,$$

where H^∞ denotes the space of boundary values of bounded analytic functions in the upper half-plane. We also treat H^∞ on the disc, and the disc algebra.

1. Introduction

Let H^∞ denote the usual Hardy space on the real line, consisting of boundary values of bounded analytic functions in the upper half-plane. We give a sufficient condition on a bijection $\phi : \mathbb{R} \to \mathbb{R}$ so that $\mathrm{Re}\, H^\infty = \mathrm{Re}\, H^\infty \circ \phi$. We derive a similar result on the unit circle S. A corollary for the disc algebra improves upon a result of O'Connell. He proved [5, Theorem 3] that if $\phi : S \to S$ is a C^2 diffeomorphism, then $\mathrm{Re}\, A = \mathrm{Re}\, A \circ \phi$. We reduce the smoothness required of ϕ to approximately C^1. We conjecture that the result holds for bi-Lipschitzian ϕ, and we give two equivalent forms of this conjecture. For related results see [1], [4], [5], [7] and [8].

2. Results

(2.1) For a locally-integrable function $f : \mathbb{R} \to \mathbb{R}$, and a closed interval I, let

$$f_I = \frac{1}{|I|}\int_I f\, dx$$

denote the mean value of f on I.

Let $E \subset \mathbb{C}$. A bijection $\phi : E \to E$ is *bi-Lipschitzian* if there exists a constant $\kappa > 0$ such that

$$\kappa^{-1}\, |x-y| \leqslant |\phi(x) - \phi(y)| \leqslant \kappa\, |x-y| \qquad \textbf{(2.1.1)}$$

for all $x, y \in E$.

Let $\phi : \mathbb{R} \to \mathbb{R}$ be bi-Lipschitzian, with inverse function $\psi = \phi^{-1}$. Then ϕ and ψ are absolutely continuous, with L^∞ derivatives. Consider the following seminorm:

$$\eta(\phi) = \sup_{a \in \mathbb{R}} \int_{\mathbb{R}} \left| \frac{\psi'(t) - \psi'_{[a,t]}}{\psi(t) - \psi(a)} \right| dt,$$

which may be $+\infty$. We have

$$\eta(\phi) \leqslant \kappa\{\eta_1(\phi) + \eta_2(\phi)\},$$

where

$$\eta_1(\phi) = \sup_{a\in\mathbb{R}} \int_{|t-a|\leq 1} \left| \frac{\psi'(t) - \psi'_{[a,t]}}{t-a} \right| dt,$$

$$\eta_2(\phi) = \sup_{a\in\mathbb{R}} \int_{|t-a|\geq 1} \left| \frac{\psi'(t) - \psi'_{[a,t]}}{t-a} \right| dt.$$

The quantity $\eta_1(\phi)$ is a kind of mean modulus of continuity of ψ'. For instance, if ψ' is uniformly Dini continuous, then $\eta_1(\phi) < \infty$. In particular, this holds if $\psi' \varepsilon$ Lip α for any $\alpha > 0$. The condition $\eta_2(\phi) < \infty$ states roughly that ϕ is approximately a translation near ∞.

(2.2) Theorem. *Let $\phi : \mathbb{R} \to \mathbb{R}$ be bi-Lipschitzian with $\eta(\phi) < \infty$: Then*

$$\operatorname{Re} H^\infty = \operatorname{Re} H^\infty \circ \phi.$$

Proof. Without loss of generality we take $\phi(0) = 0$. Choose $\kappa > 0$ so that (2.1.1) holds for all $x, y \in \mathbb{R}$. Fix $f = u + iv$ in H^∞. Let $w = H(u \circ \phi)$, where H is the Hilbert transform, that is

$$w(x) = \frac{1}{\pi} \int u(\phi(t)) \left\{ \frac{1}{t-x} - \frac{1}{t} \right\} dt,$$

where the integral is the Cauchy principal value at the singularities of the integrand (that is $t = x$ and $t = 0$). We wish to show that $w \in L^\infty$, since this will give $u \circ \phi + iw \in H^\infty$. It suffices to show that $w \circ \psi \in L^\infty$. Fix $a \in \mathbb{R}$. Then by change of variables

$$w \circ \psi(a) = \frac{1}{\pi} \int_{\mathbb{R}} u(t) \left\{ \frac{1}{\psi(t) - \psi(a)} - \frac{1}{\psi(t)} \right\} \psi'(t)\, dt, \tag{2.2.1}$$

where the integral is the limit as $r \downarrow 0$ and $s \downarrow 0$ of the integrals over

$$\mathbb{R} \sim \phi[\psi(a) - r, \psi(a) + r]) \sim \phi([-s, s]).$$

This is not a Cauchy principal value, in general, since the excised intervals are not symmetrical about the singular points. We need to estimate the difference between (2.2.1) and the Cauchy principal value integral. An upper bound for this difference is

$$\limsup_{r\downarrow 0} \{L(a, r) + L(0, r)\}$$

where

$$L(a, r) = \frac{1}{\pi} \int_{J(r)} \left| \frac{u(t)\psi'(t)}{\psi(t) - \psi(a)} \right| dt,$$

where $J(r)$ denotes the interval between $\phi(\psi(a) + r)$ and $2a - \phi(\psi(a) - r)$.

Clearly,

$$L(a, r) \leqslant \frac{\kappa^2 \|u\|_\infty}{\pi} \cdot \left| \frac{\phi(\psi(a) + r) - 2a + \phi(\psi(a) - r)}{r} \right|$$

$$= \frac{\kappa^2 \|u\|_\infty}{\pi} \cdot \left| \phi'_{[\psi(a), \psi(a)+r]} - \phi'_{[\psi(a)-r, \psi(a)]} \right|$$

$$\leqslant \frac{2\kappa^3 \|u\|_\infty}{\pi}.$$

Now since $v - v(0)$ is the Hilbert transform of u, we have

$$v(a) - v(0) = \frac{1}{\pi} \int_{\mathbb{R}} u(t) \left\{ \frac{1}{t-a} - \frac{1}{t} \right\} dt,$$

where the integral is the Cauchy principal value.

Thus

$$|w \circ \psi(a) - v(a) + v(0)| \leqslant \frac{4\kappa^3 \|u\|_\infty}{\pi} + \frac{\|u\|_\infty}{\pi} . \{I(a) + I(0)\},$$

where

$$I(a) = \int_{\mathbb{R}} \left| \frac{\psi'(t)}{\psi(t) - \psi(a)} - \frac{1}{t-a} \right| dt$$

$$= \int_{\mathbb{R}} \left| \frac{\psi'(t) - \psi'_{[a,t]}}{\psi(t) - \psi(a)} \right| dt$$

$$\leqslant \eta(\phi).$$

Hence

$$|w \circ \psi(a)| \leqslant 2 \|v\|^\infty + \left\{ \frac{4\kappa^3 + 2\eta(\phi)}{\pi} \right\} . \|u\|_\infty .$$

The result follows.

(2.3) Let $H^\infty(D)$ denote the Hardy space of bounded analytic functions on the open unit disc, regarded as a space of periodic L^∞ functions on $\mathbb{R}$. Then we obtain the following analogue.

Theorem. *Let $\phi : \mathbb{R} \to \mathbb{R}$ be a periodic bi-Lipschitzian map with period 2π, such that $\eta_1(\phi) < \infty$. Then*

$$\operatorname{Re} H^\infty(D) = \operatorname{Re} H^\infty(D) \circ \phi.$$

PROOF. In this context, the conjugation operator takes a function $u \in L^\infty(S)$ to the function $\tilde{u}$, given by

$$\tilde{u}(\theta) = \frac{1}{2\pi} \int_{\theta-\pi}^{\theta+\pi} \frac{u(\chi)}{\tan \frac{1}{2}(\chi - \theta)} d\chi,$$

where the integral is a Cauchy principal value [9, p. 131].

We obtain

$$\|u \circ \phi\|_\infty \leqq \kappa(\phi) \{\|u\|_\infty + \|\tilde{u}\|_\infty\}, \tag{2.3.1.}$$

where $\kappa(\phi)$ is a constant depending on $\eta_1(\phi)$ and the constant κ of (2.1.1). The core of the estimate goes as in (2.2); the function

$$\frac{2}{\chi - \theta} - \frac{1}{\tan \frac{1}{2}(\chi - \theta)}$$

is well-behaved, so estimating $u \circ \phi$ comes down to estimating the principal value integral

$$\int_{\theta-\pi}^{\theta+\pi} \frac{u(\chi)}{\psi(\chi) - \psi(\theta)} \cdot \psi'(\chi)\, d\chi,$$

where ϕ is the inverse function of ϕ. We can get away with $\eta_1(\phi) < \infty$ instead of $\eta(\phi) < \infty$, since the integrals are over intervals of bounded length.

(2.4) Corollary. *Let ϕ be as in* (2.3). *Then*

$$\operatorname{Re} A = \operatorname{Re} A \circ \phi,$$

where A is the disc algebra.

Proof. A is the algebra of continuous functions in $H^\infty(D)$. Let $f = u + iv$ belong to A. Choose a sequence of continuously-differentiable functions $f_n \in A$, converging uniformly to f (for instance, take the f_n to be polynomials). Let $f_n = u_n + iv_n$. Then $u_n \circ \phi$ has a continuous harmonic conjugate $u_n \circ \phi$, since the Hilbert transform of a continuously-differentiable function is continuous. By the estimate (2.3.1) the functions $u_n \circ \phi$ converge uniformly to $u \circ \phi$, hence $u \circ \phi$ is continuous, hence $u \circ \phi \in \operatorname{Re} A$.

3. Some questions

(3.1) For $1 < p < \infty$, the Hilbert transform H is bounded on $L^p(\mathbb{R})$ [6], hence $\operatorname{Re} H^p = \operatorname{Re} L^p$. Thus $\operatorname{Re} H^p = \operatorname{Re} H^p \circ \phi$ if and only if ϕ is bi-Lipschitzian. Incidentally, we use a slightly different H in this context, namely

$$Hf = \frac{1}{\pi} \int \frac{f(t)}{t - x}\, dt.$$

This form is appropriate when working with spaces which do not contain the constants.

For $0 < \alpha < 1$, the Hilbert transform is bounded on Lip $(\alpha, \mathbb{R})$ [6]. Hence, if A^α denotes the space of Lip α analytic functions on the upper half-plane, then we have $\operatorname{Re} A^\alpha = \operatorname{Re} A^\alpha \circ \phi$ if and only if ϕ is bi-Lipschitzian.

A similar result holds for the spaces of analytic functions with BMO or VMO boundary values, since BMO and VMO are invariant under the Hilbert transform, and are composition-invariant only under bi-Lipschitzian functions.

What makes these results work is not so much that the spaces are invariant under H, but that the image under H has a reasonable description. We formalise this by the following definition.

Let B be a Banach space of measurable functions on $\mathbb{R}$. We say that *B admits a metric characterisation*, or $B \in AMC$, if the map

$$\phi_{\#} : \begin{cases} B \to B \\ f \to f \circ \phi \end{cases}$$

is bicontinuous for every bi-Lipschitzian ϕ.

Obviously, $L^p(\mathbb{R})$ $(1 \leqq p \leqq \infty)$, Lip $(\alpha, \mathbb{R})$ $(0 < \alpha \leqq 1)$, BMO and VMO admit metric characterisations. Also, the space of bounded continuous functions, and the weighted L^p spaces with weights $1 + |x|^\alpha$ admit metric characterisations. However the Fourier transform does not preserve AMC. More precisely, if $(B, \|\cdot\|) \in AMC$, then the space

$$\hat{B} = \{\hat{f} : f \in B\},$$

with the norm $\|\hat{f}\| = \|f\|$, does not necessarily admit a metric characterisation. For instance, the Sobolev space $W^{2,2}$ of L^2 functions f such that f' and f'' also belong to L^2 does not admit a metric characterisation, whereas $\hat{W}^{2,2}$ is the weighted L^2 space, $L^2(1 + |x|^2)$ [3, Chapter II]. For much the same reason, not all Fourier multipliers preserve AMC. What if the multiplier is unimodular? In particular, what about the Hilbert transform?

PROBLEM. *Does $B \in AMC$ imply $HB \in AMC$?*

(3.2) For a bi-Lipschitzian $\phi : \mathbb{R} \to \mathbb{R}$, define the associated Calderon-type operator C_ϕ by

$$(C_\phi f)(x) = \frac{1}{\pi} \int \frac{f(t)}{\phi(t) - \phi(x)}\, d\phi(t),$$

where the integral is a suitable principal value (cf. (2.2)). This is similar in form to an operator considered in [2]. We have the identity

$$HC_\phi f = H\phi_{\#} H\psi_{\#} f$$

for any function f for which either side makes sense. This shows that if $B \in AMC$, then the transformed space HB admits a metric characterization if and only if the operator $HC_\phi : B \to B$ is continuous for every bi-Lipschitzian ϕ.

O'Connell [5, Theorem 2] proved that ϕ must be absolutely continuous if Re A = Re $A \circ \phi$. The natural guess for a sharp condition on ϕ for Re H = Re $H^\infty \circ \phi$ is that ϕ be bi-Lipschitzian. In view of the foregoing remarks we ask the following:

PROBLEM. Are the following equivalent conditions true:

(1) Re H^∞ admits a metric characterisation;

(2) HL^∞ admits a metric characterisation;

(3) $HC_\phi : L^\infty \to L^\infty$ is continuous for every bi-Lipschitzian ϕ?

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Added in proof: The answer to both the above problems is no. Details will appear later.