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A CHARACTERISATION OF HARMONIC FUNCTIONS

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ABSTRACT

It is well-known that harmonic functions are characterised by a mean-value property. We prove that a weaker approximate mean-value property suffices.

Let U be an open set contained in Euclidean space $\mathbb{R}^n$, and let f be a real-valued continuous function defined on U. For $a \in \mathbb{R}^n$ and $r > 0$, let

$$B(a, r) = \{x \in \mathbb{R}^n : |x - a| \leqslant r\},$$
$$\alpha_n = \mathscr{L}^n B(0, 1),$$
$$M(f, a, r) = \frac{1}{\alpha_n r^n} \int_{B(a, r)} f(x)\, d\mathscr{L}^n(x),$$

where $\mathscr{L}^n$ denotes n-dimensional Lebesgue measure. It is well-known that f is harmonic on U if and only if it enjoys the "mean value property":

$$f(a) = M(f, a, r)$$

whenever $B(a, r) \subset U$. The object of this paper is to show that an apparently weaker condition suffices to guarantee harmonicity.

Theorem *Suppose f is a continuous real-valued function on the open set $U \subset \mathbb{R}^n$, and*

$$\lim_{r \downarrow 0} \frac{M(f, a, r) - f(a)}{r^2} = 0 \tag{1}$$

for each $a \in U$. Then f is harmonic on U.

It is worth remarking that, in the case $n = 1$, this result improves upon a theorem of H. A. Schwartz. His result [3, p. 37, Theorem I] is as follows:

Suppose f is a continuous real-valued function on the open interval (c, d), and

$$\lim_{a \downarrow 0} \frac{f(a + r) - 2f(a) + f(a - r)}{r^2} = 0 \tag{2}$$

for each $a \in (c, d)$. Then f is linear.

It is clear that the Schwartz condition (2) implies our condition (1), so his result is a corollary of ours. Our result lies somewhat deeper, since the proof involves measure theory.

Proof of Theorem. Suppose f satisfies (1) for each $a \in U$. Fix an open disc D with clos $D \subset U$. Then f is bounded on D, hence the least superharmonic majorant g and the greatest subharmonic minorant h of f on D are bounded continuous functions on D. We will prove that g and h are harmonic on D, from which it follows that f is harmonic on D.

Since g is continuous on D, the set

$$E = \{x \in D : f(x) = g(x)\}$$

is relatively-closed in D. On the open set $D \sim E$, g is harmonic, so that

$$\lim_{r \downarrow 0} \frac{M(g, a, r) - g(a)}{r^2} = 0 \tag{3}$$

for each $a \in D \sim E$. On the other hand, if $a \in E$ and $B(a, r) \subset D$, then

$$\begin{aligned} 0 &\leqslant g(a) - M(g, a, r) \\ &= f(a) - M(g, a, r) \\ &\leqslant f(a) - M(f, a, r), \end{aligned}$$

so that (3) holds for $a \in E$ also, in view of (1). Thus (3) holds for all $a \in D$.

By a theorem of F. Riesz [2, p. 116; 4, p. 119], g may be written as a sum $k + p$, where k is a function harmonic on D and p is the *potential* of a finite positive Borel-regular measure μ supported on E, i.e.

$$\begin{aligned} p(x) &= -\int |y - x| \, d\mu(y), && \text{if } n = 1, \\ p(x) &= -\int \log |y - x| \, d\mu(y), && \text{if } n = 2, \text{ and} \\ p(x) &= \int |y - x|^{2-n} \, d\mu(y), && \text{if } n \geqslant 3, \end{aligned}$$

whenever $x \in D$. Since k has the mean-value property, (3) implies that

$$\lim_{r \downarrow 0} \frac{p(a) - M(p, a, r)}{r^2} = 0 \tag{4}$$

whenever $a \in D$.

At this point, we have to consider separately the cases $n = 1$, $n = 2$, and $n \geqslant 3$. We give the details for the case $n \geqslant 3$. The other cases are more or less analogous.

If $B(a, 2r) \subset D$, we have

$$\begin{aligned}
&\alpha_n 2^n r^n \{p(a) - M(p, a, 2r)\} \\
&= \int_{B(a,2r)} \{p(a) - p(x)\}\, d\mathscr{L}^n(x) \\
&= \int_{B(a,2r)} \int_D \{|y-a|^{2-n} - |y-x|^{2-n}\}\, d\mu(y)\, d\mathscr{L}^n(x) \\
&= \int_D \int_{B(a,2r)} \{|y-a|^{2-n} - |y-x|^{2-n}\}\, d\mathscr{L}^n(x)\, d\mu(y).
\end{aligned}$$

Now the function $|y-x|^{2-n}$ is superharmonic in x for each fixed y, so that

$$\int_{B(a,2r)} \{|y-a|^{2-n} - |y-x|^{2-n}\}\, d\mathscr{L}^n(x) \geqslant 0,$$

thus

$$\begin{aligned}
&\alpha_n 2^n r^n \{p(a) - M(p, a, 2r)\} \\
&\geqslant \int_{B(a,r)} \int_{B(a,2r)} \{|y-a|^{2-n} - |y-x|^{2-n}\}\, d\mathscr{L}^n(x)\, d\mu(y) \\
&\qquad = \int_{B(a,r)} \left\{ \frac{\alpha_n 2^n r^n}{|y-a|^{n-2}} - \int_{B(a,2r)} \frac{d\mathscr{L}^n(x)}{|y-x|^{n-2}} \right\} d\mu(y).
\end{aligned}$$

Let

$$A = B(a, 2r) \cap B(y, 2r),$$
$$B = B(a, 2r) \sim B(y, 2r).$$

Then by symmetry,

$$\int_A \frac{d\mathscr{L}^n(x)}{|y-x|^{n-2}} = \int_A \frac{d\mathscr{L}^n(x)}{|a-x|^{n-2}}.$$

Also, for $x \in B$ we have

$$|x-a| \leqslant 2r < |x-y|,$$

hence

$$\int_B \frac{d\mathscr{L}^n(x)}{|y-x|^{n-2}} \leqslant \int_B \frac{d\mathscr{L}^n(x)}{|a-x|^{n-2}}.$$

Thus, since $B(a, 2r) = A \cup B$, we have

$$\int_{B(a,2r)} \frac{d\mathscr{L}^n(x)}{|y-x|^{n-2}} \leqslant \int_{B(a,2r)} \frac{d\mathscr{L}^n(x)}{|a-x|^{n-2}}$$

$$= 2n\alpha_n r^2,$$

so that

$$\begin{aligned} & p(a) - M(p, a, 2r) \\ & \geqslant \int_{B(a,r)} \left\{ \frac{1}{|y-a|^{n-2}} - \frac{n}{2^{n-1}r^{n-2}} \right\} d\mu(y) \\ & > r^{2-n} \left\{ 1 - \frac{n}{2^{n-1}} \right\} \mu B(a, r) \\ & \geqslant \frac{\mu B(a, r)}{4r^{n-2}}. \end{aligned}$$

Hence, by (4), for each $a \varepsilon D$ we have

$$\lim_{r \downarrow 0} \frac{\mu B(a, r)}{r^n} = 0.$$

The density theorem [1, p. 181, (2.10.19)(1)] now implies that $\mu = 0$, hence $g = k$ is harmonic on D.

The proof that h is harmonic on D is similar. This completes the proof.

We remark that the exponent 2 cannot be reduced in condition (1), since

$$\lim_{r \to 0} \frac{M(f, a, r) - f(a)}{r^\beta} = 0$$

whenever $\beta < 2$ and f is twice continuously differentiable at a.

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