# ANALYTIC CAPACITY AND EQUICONTINUITY

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BLMS 10 (1978) 276-9

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In this paper we prove the following conjectures of Wang: Let $\phi(r)$ be a positive nondecreasing admissible function, let $a$ be a point of a compact set $X \subset \mathbb{C}$, let $A_n = \{z \in \mathbb{C} : 2^{-n} < |z-a| < 2^{-n+1}\}$ and suppose that

$$\sum_{n=1}^{\infty} 2^n \phi(2^{-n})^{-1} \gamma(A_n \sim X) < \infty,$$

where $\gamma$ denotes analytic capacity. Then (1) there exists a representing measure $\mu$ for $a$ on $R(X)$ such that $\int \phi(|z-a|)^{-1} \, d|\mu|(z) < \infty$, and (2) $R(X)$ admits $\phi$ as a modulus of approximate equicontinuity at $a$.

Let $X$ be a compact subset of the complex plane $\mathbb{C}$. We denote by $\mathscr{R}(X)$ the space of all functions, continuous on $\mathbb{C}$ and analytic on a neighbourhood of $X$. We denote by $R(X)$ the closure of $\mathscr{R}(X)$ with respect to the uniform norm on $X$:

$$\|f\|_X = \sup_X |f|.$$

The uniform algebra $R(X)$ has been studied intensively [**1**, **2**, **3**]. Apart from the natural questions of uniform rational approximation, interest has focused on $R(X)$ as a source of counterexamples in the theory of Banach algebras. Indeed, the pathological behaviour of $R(X)$ is so extraordinary that any positive result seems impressive. The most striking early result is Browder's metric density theorem [**1**, p. 177] which states that *at any non-peak point $a \in X$ of $R(X)$ the unit ball of $R(X)$ is approximately equicontinuous*, i.e., for each $\varepsilon > 0$ the set

$$\{z \in X : |f(z) - f(a)| < \varepsilon \text{ for all } f \in R(X) \text{ with } \|f\|_X \leqslant 1\}$$

has full area density at $a$. This result was strengthened in various ways by the author [**5**, **6**, **7**], Wang [**9**, **10**], Øksendal [**8**] and Hayashi [**4**]. In particular, Wang [**10**] showed that at almost all nonpeak points, and for all $\alpha$ less than 1, all the functions belonging to the unit ball of $R(X)$ satisfy a single Hölder condition of order $\alpha$ on a set of full density.

In his paper [**10**], Wang formulated three conditions, each of which might be interpreted as saying that the functions belonging to $R(X)$ are of a certain degree of smoothness at the point $a$. To describe his conditions, we need some notation and terminology.

An *admissible function* is a positive nondecreasing function $\phi(r)$ on the interval $(0, \infty)$ such that the *associated function* $\psi(r) = r/\phi(r)$ is also nondecreasing, with $\psi(0+) = 0$. We denote the (inner) *analytic capacity* of an open set $\mathscr{V}$ by $\gamma(\mathscr{V})$ [**2**, p. 196], and we set

$$A_n(a) = \{z \in \mathbb{C} : 2^{-n} < |z-a| < 2^{-n+1}\}$$

Received 12 November, 1977

[BULL. LONDON MATH. SOC., 10 (1978), 276–279]

whenever $a \in \mathbb{C}$. A finite Borel—regular measure $\mu$ on $X$ is a (complex) *representing measure* for $a$ on $R(X)$ if $f(a) = \int f du$ whenever $f \in R(X)$. We denote the *total variation measure* of $\mu$ by $|\mu|$, and the *total variation*, $\int d|\mu|$, of $\mu$ by $\|\mu\|$. For a function $f$ analytic at $a$, and a nonnegative integer $p$, we let

$$R_a{}^p f(z) = f(z) - \sum_{j=0}^{p} \frac{f^{(j)}(a)}{j!}(z-a)^j$$

be the " error " at $z$ of the *p-th* degree Taylor polynomial of $f$ about $a$.

Let $X$ be a compact subset of the plane, let $a \in X$, let $p$ be a nonnegative integer, and let $\phi(r)$ be an admissible function. Wang's conditions are as follows:

(A) *For each* $\varepsilon > 0$, *the set*

$$\{z \in X : |R_a{}^p f(z)| \leqslant \varepsilon\phi(|z-a|)|z-a|^p \|f\|_X \text{ for all } f \in \mathscr{R}(X)\}$$

*has full area density at a.*

(B) *There exists a representing measure* $\mu$ *for* $a$ *on* $R(X)$ *such that* $\mu\{a\} = 0$ *and*

$$\int \frac{d|\mu|(z)}{|z-a|^p \phi(|z-a|)} < \infty .$$

(C) *The series*

$$\sum_{n=1}^{\infty} \frac{2^{(p+1)n}\gamma(A_n(a) \sim X)}{\phi(2^{-n})}$$

*converges.*

To date, the known relations between these conditions are as follows.

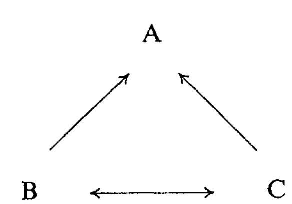
(1) The three conditions are equivalent for $\phi = 1$ (Melnikov, Hallstrom, Browder, Wilken, Wang).

(2) B implies C (O'Farrell, Wang).

(3) B implies A (Wang).

(4) A does not imply C, in general (Wang). Hence A does not imply B.

Wang conjectured that C implies A and B. The purpose of the present paper is to prove this conjecture and round off this subject. The final result is summarized in the diagram:




Since B implies A, it suffices to prove that C implies B. We shall in fact prove this implication for arbitrary nondecreasing $\phi$. This allows us to treat the case $p = 0$ without loss of generality.

THEOREM. *Let $\phi(r)$ be a positive nondecreasing function on $(0, \infty)$. Let $X$ be a compact subset of the plane, and let $a \in X$. Suppose*

$$\sum_{n=1}^{\infty} \frac{2^n \gamma(A_n(a) \sim X)}{\phi(2^{-n})} < \infty .$$

*Then $a$ has a representing measure $\mu$ on $R(X)$ such that $\mu\{a\} = 0$ and*

$$\int \frac{d|\mu|(z)}{\phi(|z-a|)} < \infty .$$

*Proof.* We may take $a = 0$ without loss of generality. We abbreviate $A_n = A_n(0)$. Let $f$ be continuous on $\text{clos}A_n$ and analytic off $A_n \sim X$.

Define

$$T_n f = \frac{-1}{2\pi i} \int_{\text{bdy}A_n} \frac{f(z)}{z} dz$$

where $\text{bdy}A_n$ is given the usual orientation, leaving $A_n$ on the left. By a theorem of Melnikov [2, (viii. 12.6), p. 232],

$$|T_n f| \leqslant C 2^n \gamma(A_n \sim X) \|f\|_{A_n}$$

where $C$ is a certain universal constant. By the Hahn-Banach theorem and the Riesz representation theorem, there exists a Borel—regular measure $\mu_n$ on $\text{clos}A_n$ such that

$$\|\mu_n\| \leqslant C 2^n \gamma(A_n \sim X)$$

and

$$T_n f = \int f d\mu_n$$

whenever $f$ is continuous on $\text{clos}A_n$ and analytic off $A_n \sim X$. Since $\Sigma 2^n \gamma(A_n \sim X) < \infty$ we may define a finite measure $\nu$ by setting

$$\nu = \sum_1^{\infty} \mu_n .$$

Fix $f \in \mathscr{R}(X)$. Since $f$ is analytic on a neighbourhood of $a$, there exists a positive integer $N$ such that $f$ is analytic inside and on the circle $|z| = 2^{-N}$. Thus

$$\begin{aligned} f(0) &= \frac{1}{2\pi i} \int_{|z|=2^{-N}} \frac{f(z)}{z} dz \\ &= \frac{1}{2\pi i} \int_{|z|=1} \frac{f(z)}{z} dz - \sum_{n=1}^{\infty} \frac{1}{2\pi i} \int_{\text{bdy}A_n} \frac{f(z)}{z} dz \\ &= \frac{1}{2\pi i} \int_{|z|=1} \frac{f(z)}{z} dz + \int f d\nu . \end{aligned}$$

Let $\theta$ denote the measure $dz/2\pi iz$ on the unit circle, and let $\mu = \theta + \nu$. Then

$$f(0) = \int f d\mu \tag{1}$$

whenever $f \in \mathscr{R}(X)$. If $D$ is a closed disc disjoint from $X$, then $\mathscr{R}(X)$ contains every continuous function which vanishes off $D$. Hence $\mu$ has no mass on the interior of $D$. Thus $\mu$ is supported on $X$. It follows by continuity that (1) holds for all $f \in R(X)$. Thus $\mu$ represents 0 on $R(X)$. Finally, $\mu\{0\} = 0$ and

$$\begin{aligned}\int \frac{d|\mu|(z)}{\phi(|z|)} &\leqslant \int \frac{d\theta(z)}{\phi(|z|)} + \int \frac{d|\nu|(z)}{\phi(|z|)} \\ &\leqslant \frac{1}{\phi(1)} + \sum_{n=1}^{\infty} \frac{\|\mu_n\|}{\phi(2^{-n})} \\ &\leqslant \frac{1}{\phi(1)} + \sum_{n=1}^{\infty} \frac{C2^n\gamma(A_n \sim X)}{\phi(2^{-n})} \\ &< \infty .\end{aligned}$$

The proof is complete.

As usual, the above theorem carries over to all T-invariant algebras.

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