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RATIONAL APPROXIMATION IN LIPSCHITZ NORMS—I

By A. G. O'Farrell

St Patrick's College, Maynooth

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Abstract

Let E be a closed subset of the complex plane $\mathbb{C}$. This paper concerns the closure in Lip (α, E) for $0 < \alpha \leqslant 1$, of the space of rational functions with poles off E. It may be regarded as a continuation of the author's work in [4, 5, 6, 7]. The main result is that the tale of Lip 1 rational approximation is the same as that of C^1 rational approximation. We also give an improvement of a result of Browder, concerning the functions that satisfy the Cauchy-Riemann equations at the points of a compact set.

Let C^1 denote the space of bounded continuous complex-valued functions on $\mathbb{C}$ with bounded continuous first partial derivatives. For $f \in C^1$, let

$$\|f\|_{C^1} = \sup|f| + \sup|\nabla f|$$

where $\nabla f = (f_x, f_y)$ is the gradient of f, and let

$$\frac{df}{dz} = f_x - if_y$$

$$\frac{df}{d\bar{z}} = f_x + if_y.$$

Let E be a closed subset of $\mathbb{C}$, and let $0 < \alpha \leqslant 1$. For $f : E \to \mathbb{C}$, let

$$\|f\|_{\alpha,E} = \sup_E |f| + \inf\{\kappa > 0 : |f(a) - f(b)| \leqslant \kappa\, |a-b|^\alpha, \text{ for all } a, b \in E\}.$$

Let

$$\text{Lip}(\alpha, E) = \{f : \|f\|_{\alpha E} < +\infty\},$$
$$\text{Lip}\,\alpha = \text{Lip}(\alpha, \mathbb{C}).$$

Let $\mathscr{R}(E)$ denote the set of functions $f \in C^1$ such that f agrees with a rational function on some neighbourhood of E. Let E^d denote the derived set, or set of accumulation points, of E.

Theorem 1. Let E be a closed subset of the complex plane, and let f be a function in the class Lip $(1, E)$. Then the following conditions are equivalent:

(A) f is the limit in Lip $(1, E)$ norm of a sequence of elements of $\mathscr{R}(E)$.

(B) f has an extension f^* in the class C^1, and f^* is the limit in C^1 norm of a sequence of elements of $\mathscr{R}(E)$.

Proof. It is easy to see that (*B*) implies (*A*). To see that (*A*) implies (*B*), suppose (*A*) holds, and choose a sequence of functions $f_n \in \mathscr{R}(E)$ such that

$$\|f_n - f\|_{1,E} \to 0.$$

The complex derivatives df_n/dz form a Cauchy sequence (in the uniform norm on E^d) of continuous functions on E^d, and hence they converge, uniformly on compact subsets of E^d to a continuous function g. Extend g to a continuous function (also denoted g) on E. It is easy to check that, given $\varepsilon > 0$ and C compact, $C \subset E$, there exists $\delta > 0$ such that

$$|f(a) - f(b) - g(b)(a-b)| < \varepsilon\, |a-b|$$

whenever $a,\ b \in C$, $|a-b| < \delta$. By Whitney's extension theorem [3, p. 225], there exists an extension f^* of f in C^1, such that $df^*/dz = g$ on E, and $df^*/d\bar{z} = 0$ on E.

Let $\varepsilon > 0$ be given. There exists an integer N such that

$$\|f_n - f\|_{1,E} < \varepsilon$$

whenever $n > N$. Fix $n > N$. Then

$$\left[\frac{df_n}{dz} - \frac{df^*}{dz} \right] < \varepsilon$$

on E^d. Thus

$$|\nabla f_n - \nabla f^*| < \varepsilon \tag{1}$$

on E^d, since $df_n/d\bar{z} = df^*/d\bar{z} = 0$ on E^d. By continuity, there exists a closed neighbourhood $U = U_n$ of E^d such that (1) holds on U. Now $E \sim U$ is locally-finite, so we may modify f_n off U, to obtain a function $g_n \in \mathscr{R}(E)$ and a closed neighbourhood V of E such that

$$\|g_n - f^*\|_{1,V} < 2\varepsilon$$

(indeed, g_n may be taken as constant on a neighbourhood of each point of $E \sim U$). By Kirszbraun's theorem [3, p. 201] about extending Lip 1 functions, we may modify g_n off V to obtain a function $h_n \in \mathscr{R}(E)$ such that

$$\|h_n - f^*\|_{1,\mathbb{C}} < 4\varepsilon,$$

and hence

$$\|h_n - f^*\|_{C^1} < 8\varepsilon.$$

This completes the proof.

In order for a function in C^1 to be approximable in C^1 norm by functions in $\mathscr{R}(E)$, it must satisfy the Cauchy-Riemann equation $df/d\bar{z} = 0$ at each point of E. I do not believe this condition is sufficient, but I know of no counterexample. For a reasonably wide class of compact sets E the condition *is* sufficient, as may be seen by using the constructive methods of [5]. Our second theorem shows that this condition implies that the function is approximable by rationals in Lip α for all α *less* than 1. The proof uses the technique of the Cauchy transform of distributions. We refer the reader to [6] for the basic properties of this operator.

Theorem 2. Let E be a closed subset of $\mathbb{C}$. Suppose $f \in C^1$ and $df/d\bar{z} = 0$ on E. Then for each α, $0 < \alpha < 1$, the function f belongs to the closure of $\mathscr{R}(E)$ in Lip (α, E).

PROOF. Let T be a continuous linear functional on Lip α that annihilates $\mathscr{R}(E)$. Then the Cauchy transform $\hat{T}$ is a measure [6, Lemma 5] supported on E. Hence, $Tf = \hat{T}(df/d\bar{z}) = 0$. By the separation theorem, f belongs to the closure of $\mathscr{R}(E)$ in Lip α. *A fortiori*, f belongs to the closure of $\mathscr{R}(E)$ in Lip (α, E).

Remarks.

1. In view of the fact that Lip α convergence implies uniform convergence, Theorem 2 strengthens [1, Corollary (3.2.2)]. Consequently, it permits strengthening the conclusion of [2, (1.5) (iv)] and [9].

2. Dales and Davie [2], and Rubel [9] provide conditions on a function $f: E \to \mathbb{C}$ which guarantee the existence of a C^1 extension. (Dales and Davie consider compact perfect sets in the plane, and Rubel considers compact sets in the plane. Both results involve complex derivatives.) For a more comprehensive result of this kind, see [8].

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REFERENCES

[1] BROWDER, A. 1969 *Introduction to function algebras*. Benjamin.

[2] DALES, H. G. and DAVIE, A. M. 1973 Quasianalytic Banach function algebras. *J. funct. Anal.* **13**, 28–50.

[3] FEDERER, H. 1969 *Geometric Measure Theory*. Springer.

[4] O'FARRELL, A. G. 1975 Lip 1 rational approximation. *J. Lond. math. Soc.* **11**, 159–164.

[5] O'FARRELL, A. G. 1977 Hausdorff content and rational approximation in fractional Lipschitz norms. *Trans. Am. math. Soc.* **228**, 187–206.

[6] O'FARRELL, A. G. 1975 Annihilators of rational modules. *J. funct. Anal.* **19**, 373–389.

[7] O'FARRELL, A. G. 1975 Localness of certain Banach modules. *Indiana U. math. J.* **24**. 1135–1141.

[8] O'FARRELL, A. G. 1976 Functions with smooth extensions. *Proc. R. Ir. Acad.* 76A (29), 317–320

[9] RUBEL, L. A. 1975 An extension of Runge's theorem. *Proc. Am. math. Soc.* **47**, 261–262.