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ANALYTIC CAPACITY, HÖLDER CONDITIONS, AND τ-SPIKES

BY

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ABSTRACT. We consider the uniform algebra $R(X)$, for compact $X \subset \mathbf{C}$, in relation to the condition $I_{p+\alpha} = \sum_1^\infty 2^{(p+\alpha+1)n} \gamma(A_n(x) \setminus X) < +\infty$, where $0 \le p \in \mathbf{Z}$, $0 < \alpha < 1$, γ is analytic capacity, and $A_n(x)$ is the annulus $\{z \in \mathbf{C}: 2^{-n-1} < |z-x| < 2^{-n}\}$. We introduce the notion of τ-spike for $\tau > 0$, and show that $I_{p+\alpha} = +\infty$ implies x is a $p+\alpha$-spike. If $\overset{\circ}{X}$ satisfies a cone condition at x, and $I_{p+\alpha} < +\infty$, we show that the pth derivatives of the functions in $R(X)$ satisfy a uniform Hölder condition at x for nontangential approach. The structure of the set of non-τ-spikes is examined and the results are applied to rational approximation. A geometric question is settled.

1. For a compact subset X of the Riemann sphere Σ, $R(X)$ denotes the uniform closure on X of the collection $R_0(X)$ of rational functions with poles off X. $R(X)$ is a Banach algebra with respect to the uniform norm $\|\cdot\|_X$ on X. For a positive integer p, $R(X)$ is said to admit a pth order *bounded point derivation* at a point $x \in X$ if the linear functional on $R_0(X)$ defined by $f \mapsto f^{(p)}(x)$ (=the pth derivative of f at x) extends to a continuous linear functional on $R(X)$, i.e., if

$$\sup\{|f^{(p)}(x)| : f \in R_0(X),\ \|f\|_X \le 1\} < +\infty.$$

Hallstrom [4] characterised the points of X at which pth order bounded point derivations exist in terms of analytic capacity, γ. If $U \subset \mathbf{C}$ is a bounded open set we define

$$\gamma(U) = \sup\{|f'(\infty)| : f \in R(\Sigma \setminus U),\ \|f\|_{\Sigma \setminus U} \le 1\}$$

and denote for $x \in \mathbf{C}$, $n \in \mathbf{Z}$, $r \in \mathbf{R}$,

$$A_n(x) = \{z \in \mathbf{C} : 2^{-n-1} < |z-x| < 2^{-n}\},$$
$$U(x,r) = \{z \in \mathbf{C} : |z-x| < r\},$$
$$B(x,r) = \{z \in \mathbf{C} : |z-x| \le r\}.$$

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Hallstrom's theorem. *Let X be a compact subset of $\mathbf{C}$, $x \in X$, $0 < p \in \mathbf{Z}$. Then $R(X)$ admits a pth order bounded point derivation at x if and only if*

$$\sum_{n=1}^{+\infty} 2^{(p+1)n}\gamma(A_n(x)\backslash X) < +\infty.$$

This is an extension of sorts of Mel′nikov's theorem [2] characterising the *peak points* for $R(X)$. A point $x \in X$ is said to be a peak point for $R(X)$ if there is a function $f \in R(X)$ such that $f(x) = 1$ and $|f(z)| < 1$ for every $z \in X\backslash\{x\}$.

Mel′nikov's theorem. *Let $X \subset \mathbf{C}$ be compact, $x \in X$. Then x is a peak point for $R(X)$ if and only if*

$$\sum_{n=1}^{+\infty} 2^{n}\gamma(A_n(x)\backslash X) = +\infty.$$

Thus the condition of Mel′nikov's theorem corresponds to that of Hallstrom's, with p replaced by 0. For convenience let us say that $R(X)$ admits a 0th order bounded point derivation at x if x is a nonpeak point.

A. Browder asked what might be the significance for $R(X)$ of Hallstrom's condition for nonintegral p. That is, if $0 < \lambda \in \mathbf{R}$, what does the condition

$$I_\lambda(X, x) = \sum_{n=1}^{+\infty} 2^{(\lambda+1)n}\gamma(A_n(x)\backslash X) = +\infty$$

tell us about the function-theoretic properties of $R(X)$ near x? The idea is that this condition should be related to some kind of λth derivative at x of the functions in $R(X)$.

2. For $0 \le p \in \mathbf{Z}$, the pth order *Gleason metric* d^p on X is defined by

$$d^p(x, y) = \sup\{|f^{(p)}(x) - f^{(p)}(y)| : f \in R_0(x),\ \|f\|_X \le 1\},$$

whenever $x, y \in X$. Note that $d^p(x, y)$ may be $+\infty$. This metric was studied in [7], from the point of view of determining for a point $x \in \partial X$ whether there exists a sequence of points $y_n \to x$, $y_n \in X$, $y_n \ne x$, such that $d^p(y_n, x) \to 0$. In particular, the following things are true [7, Corollary 1, Corollary 3]: *Suppose $\mathring{X}$ satisfies a cone condition at x, i.e. there is a triangle in $\mathring{X} \cup \{x\}$ with vertex at x, and Γ denotes the midline of the triangle. Let $0 \le p \in \mathbf{Z}$. Then, if $R(X)$ admits a pth order bounded point derivation at x, it follows that $d^p(y, x) \to 0$ as $y \to x$, $y \in \Gamma$. If $R(X)$ admits a $(p+1)$st order bounded point derivation at x, then there is a constant $\kappa > 0$ such that $d^p(y, x) \le \kappa|y - x|$, whenever $y \in \Gamma$.* Abbreviating $I_\lambda = I_\lambda(X, x)$, and combining these facts with the theorems of Mel′nikov and Hallstrom, we deduce that $I_p < +\infty$ implies $d^p(y, x) \to 0$ for $y \in \Gamma$, and $I_{p+1} < +\infty$

implies $d^p(y, x) \le \kappa|y - x|$ for $y \in \Gamma$, so a reasonable guess is that $I_{p+\alpha} < +\infty$ should imply a condition $d^p(y, x) \le \kappa|y - x|^\alpha$.

Theorem 1. *Suppose* $X \subset \mathbf{C}$ *is compact,* $x \in X$, $\mathring{X}$ *satisfies a cone condition at* x, Γ *is the midline of a sector* C *with vertex* x *which lies in* $\mathring{X} \cup \{x\}$, $0 \le P \in \mathbf{Z}$, $0 < \alpha < 1$, *and* $I_{p+\alpha} < +\infty$. *Then there is a constant* $\kappa > 0$ *such that*

(1) $$d^p(y, x) \le \kappa|y - x|^\alpha$$

whenever $y \in \Gamma$.

Proof. We may suppose $x = 0$, $\Gamma = [-1, 0]$, $C = \{z \in \mathbf{C}: |z| \le 1, |\arg(\pi - z)| \le \alpha\}$ for some $\alpha > 0$. Observe that it suffices to produce a κ such that (1) holds for $y \in [-\frac{1}{2}, 0]$, for given such a κ, (1) then holds with κ replaced by

$$\max\{\kappa, \sup\{d^p(y, x)|y - x|^{-\alpha}: -1 \le y \le -\tfrac{1}{2}\}\}.$$

Fix $y \in [-\frac{1}{2}, 0]$, $f \in R_0(X)$.

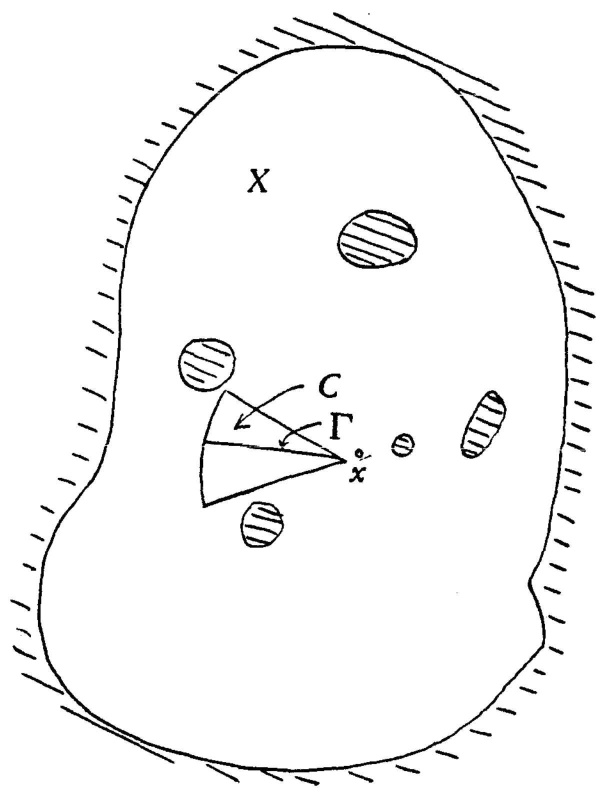

Figure 1

There exists a positive integer N such that f is analytic on $B = B(0, 2^{-N-1})$. Hence

$$f^{(p)}(y) - f^{(p)}(x) = \frac{p!}{2\pi i} \oint_{\partial(B \cup C)} f(z)\{(z-y)^{-(p+1)} - (z-x)^{-(p+1)}\} dz$$

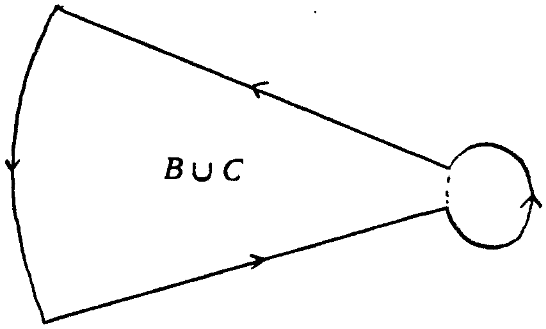

Figure 2

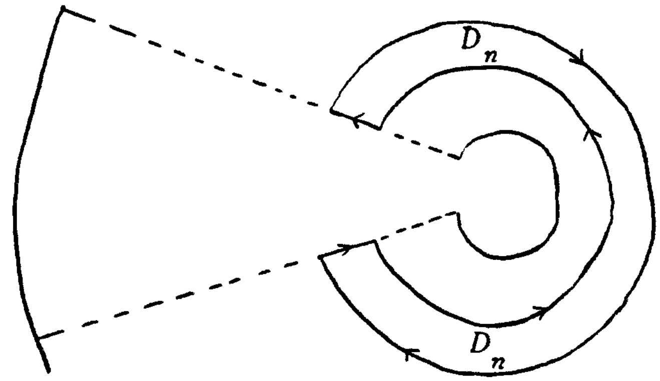

Figure 3

$$= \frac{p!}{2\pi i} \sum_{n=1}^{N} \oint_{\partial D_n} f(z)\{\cdot\} dz + \frac{p!}{2\pi i} \oint_{|z|=1} f(z)\{\cdot\} dz,$$

where $D_n = A_n(0) \setminus C$, and in the integral the orientation of ∂D_n is that which leaves D_n on the right.

Select $q \in \mathbf{Z}$, $q > 1$ such that $y \in A_q(0)$. There is a constant $\tau > 0$ such that $|\zeta - x| \le \tau |\zeta - y|$ for all $\zeta \notin C$ (and τ may be chosen independent of $y \in [-\frac{1}{2}, 0]$). Hence, if $q + 2 \le n \le N$, $\zeta \in D_n$, then

$$|\{\cdot\}| = \frac{(x-y) \sum_{m=0}^{p} \binom{p}{m} (\zeta - x)^m (\zeta - y)^{p-m}}{(\zeta - x)^{p+1} (\zeta - y)^{p+1}}$$

$$\le |x - y| \sum_{m=0}^{p} \binom{p}{m} |\zeta - x|^{m-p-1} |\zeta - y|^{-m-1}$$

$$\le |x - y|^{\alpha} \sum_{m=0}^{p} \binom{p}{m} |\zeta - x|^{m-p-1} 2^{1-\alpha} |\zeta - y|^{-m-\alpha}$$

$$\text{(since } |y - x| \le 2|\zeta - y|)$$

$$\le |x - y|^{\alpha} \sum_{m=0}^{p} \binom{p}{m} \tau^{m+\alpha} 2^{1-\alpha} |\zeta - x|^{-p-x-1}$$

$$\le |x - y|^{\alpha} \tau^{\alpha} (1 + \tau)^p 2^{1-\alpha} 2^{(p+\alpha+1)n}.$$

If $1 \le n \le q - 2$, $\zeta \in D_n$, then

$$|\{\cdot\}| \le |x - y| \sum_{m=0}^{p} \binom{p}{m} |\zeta - x|^{m-p-1} |\zeta - y|^{-m-1} \le |x - y| \sum_{m=0}^{p} \binom{p}{m} \tau^{m+1} |\zeta - x|^{-p-2}$$

$$\le |x - y| \tau (1 + \alpha)^p 2^{(p+2)n} \le |x - y|^{\alpha} \tau (1 + \tau)^p 2^{(p+\alpha+1)n}.$$

If $n = q - 1$, q, or $q + 1$, $\zeta \in D_n$, then

$$\|\{\cdot\}\| \le |x-y| \sum_{m=0}^{p} \binom{p}{m} |\zeta - x|^{m-p-1}|\zeta - y|^{-m-1} \le |x-y| \sum_{m=0}^{p} \binom{p}{m} \tau^{m+1}2^{(p+2)(q+2)}$$

$$\le |x-y|^{\alpha}\tau(1+\tau)^p 4^{p+2}2^{(p+\alpha+1)n} \le |x-y|^{\alpha}\tau(1+\tau)^p 8^{p+2}2^{(p+\alpha+1)n}.$$

Thus, taking λ to be the largest of the numbers $\tau^{\alpha}(1+\tau)^p 2^{1-\alpha}$, $8^{p+2}\tau(1+\tau)^p$, we have $\|\{\cdot\}\| \le \lambda |x-y|^{\alpha} 2^{(p+\alpha+1)n}$ wherever $1 \le n \le N$, $\zeta \in D_n$. Now, applying the Mel'nikov integral estimate [5], [9], [2] to the (pairwise similar) regions D_n, there is a constant $L > 0$ such that

$$\left| \int_{\partial D_n} g(\zeta)\, d\zeta \right| < L \|g\|_{D_n} \gamma(U \cap D_n)$$

where $g \in R(D_n \setminus U)$, $n = 1, 2, 3, \cdots$. Thus

$$|f^{(p)}(y) - f^{(p)}(x)|$$

$$\le \frac{p!}{2\pi} \sum_{n=1}^{N} \|f\|_X \lambda \cdot L \cdot |x-y|^{\alpha} 2^{(p+\alpha+1)n}\gamma(D_n \setminus X) + \frac{p!}{2\pi} \|f\|_X |x-y| 2^{2p+1}$$

$$\le \frac{p!}{2\pi} \left\{ \lambda \cdot L \sum_{n=1}^{+\infty} 2^{(p+\alpha+1)n}\gamma(A_n(0)\setminus X) + 2^{2p+\alpha} \right\} \|f\|_X |x-y|^{\alpha}.$$

Thus (1) holds with

$$\kappa = \frac{p!}{2\pi} \left\{ \lambda \cdot L \sum_{n=1}^{+\infty} 2^{(p+\alpha+1)n}\gamma(A_n(0)\setminus X) + 2^{2p+\alpha} \right\}.$$

In plain language the conclusion of Theorem 1 is that for nontangential approach to x from $\overset{\circ}{X}$, the pth derivatives of the functions in $R(X)$ satisfy a uniform Hölder condition: $|f^{(p)}(x) - f^{(p)}(y)| \le \kappa \|f\|_X |x-y|^{\alpha}$, where κ is independent of f and y.

3. Wilken [11] observed that $R(X)$ *admits a pth order bounded point derivation at x $(p \ge 1)$ if and only if x has a representing measure μ on $R(X)$ such that* $\mu^p(x) < +\infty$. (Recall that a complex Radon measure μ *represents* x on $R(x)$ if $\int f\, d\mu = f(x)$ whenever $f \in R(X)$; and for $0 < \beta \in \mathbf{R}$ the *potential of order* β, μ^{β}, of μ is the function defined by $\mu^{\beta}(z) = \int d|\mu|(\zeta)/|\zeta - z|^{\beta}$ for $z \in \mathbf{C}$; here $|\mu|$ denotes the total variation measure of μ.) This provides us with a second natural way of interpolating between p and $p+1$. In these terms we obtain a result in the opposite direction to Theorem 1, but in a more general setting.

Theorem 2. *Suppose $X \subset \mathbf{C}$ is compact, $x \in X$, $0 \le p \in \mathbf{Z}$, $0 < \alpha < 1$, and $I_{p+\alpha} = +\infty$. Then $\mu^{p+\alpha}(x) = +\infty$ whenever μ is a representing measure for x on $R(X)$.*

Proof. For convenience, suppose $\operatorname{diam} X \leq \frac{1}{2}$. There are two cases to consider.

Case 1°. $\limsup_{n\to\infty} 2^{(p+\alpha+1)n}\gamma(A_n(x)\setminus X) = 0$, so that, for some integer N_0, all the terms beyond the N_0th are bounded by 1. Fix $N_0 < N \in \mathbf{Z}$ and choose $M \geq N$, $M \in \mathbf{Z}$ such that

$$1 \leq \sum_{n=N}^{M} 2^{(p+\alpha+1)n}\gamma(A_n(x)\setminus X) \leq 2.$$

For each $n \in \mathbf{Z}$ with $N \leq n \leq M$ choose $f_n \in R(X \cup (\Sigma\setminus A_n))$ such that $\|f_n\|_\Sigma \leq 1$, $f_n(\infty) = 0$, $f_n'(\infty) > \frac{1}{2}\gamma(A_n(x)\setminus X)$. Form $g_N(z) = |z-x|^\alpha (z-x)^{p+1}\sum_{n=N}^M 2^{(p+\alpha+1)n} f_n(z)$. Then a familiar type of argument (cf. [2, p. 206]) shows that the sequence $\{g_N\}_1^\infty$ is uniformly bounded on any bounded set. Defining $h_N(z) = |z-x|^{-\alpha}(z-x)g_N(z)$, we see that $\{h_N\}_1^\infty$ is bounded on bounded sets, and since h_N is analytic on $\Sigma\setminus B(x, 2^{-N})$ we deduce that a subsequence (again denoted $\{h_N\}$) converges pointwise on $\mathbf{C}\setminus\{x\}$ to a function h which is analytic on $\mathbf{C}\setminus\{x\}$. Since h is bounded near x, h is entire. Letting $k_N(z) = (z-x)^{-p-2}h_N(z)$, we see that

$$k_n'(\infty) = \lim_{z\to\infty} (z-x)k_N(z) = \sum_{n=N}^{M} 2^{(p+\alpha+1)n} f'(\infty)$$

lies in $[1, 2^{p+\alpha}]$ for each N, hence by passing to a second subsequence we have $k_N'(\infty) \to \beta$ for some $\beta \in [1, 2^{p+\alpha}]$. Thus $\lim_{z\to\infty}(z-x)^{-p-1}h(z) = \beta$, hence $h(z) = \beta(z-x)^{p+1}$ for $z \in \mathbf{C}$, hence $g_N(z)$ tends pointwise boundedly on bounded subsets of $\mathbf{C}$ to $\beta|z-x|^\alpha(z-x)^p$.

Suppose μ is a representing measure for x on $R(X)$ with $\mu^{p+\alpha}(x) < +\infty$. Then $|z-x|^{-\alpha}(z-x)^{-p}\mu$ is a finite measure and, setting $l_N(z) = |z-x|^{-\alpha}g_N(z)$, l_N is analytic near x, $l_N \in R(X)$,

$$0 = l_N^{(p)}(x) = p!\int \frac{l_N(z)}{(z-x)^p}\,d\mu(z) = p!\int \frac{g_N(z)}{|z-x|^\alpha (z-x)^p}\,d\mu(z)$$

$$\to p!\int \beta\,d\mu(z) = \beta\cdot p!.$$

This is a contradiction.

Case 2°. $\limsup_{n\to+\infty} 2^{(p+\alpha+1)n}\gamma(A_n(x)\setminus X) > 2S > 0$. Let $\{N_i\}_1^\infty$ be a sequence of integers such that

$$2^{(p+\alpha+1)N_i}\gamma(A_{N_i}(x)\setminus X) > 2S,$$

and for each i choose $f_i \in R(X \cup (\Sigma\setminus A_{N_i}))$ such that $\|f_i\|_\Sigma \leq 1$, $f_i(\infty) = 0$,

$f_i'(\infty) = S2^{-(p+\alpha+1)N_i}$. Then, defining $g_i(z) = |z-x|^\alpha (z-x)^{p+1} 2^{(p+\alpha+1)N_i} f_i(z)$, the argument of Case 1° goes through with these new g_i's, and again we arrive at a contradiction.

4. Let us say that x is a *τ-spike* for $R(X)$ if $\mu^\tau(x) = +\infty$ whenever μ represents x on $R(X)$. A peak point is a τ-spike for every $\tau > 0$.

Corollary 1. *Suppose $\overset{\circ}{X}$ satisfies a cone condition at x, Γ is a straight line in $\overset{\circ}{X} \cup \{x\}$ which is not tangential to ∂X at x, $0 \le p \in \mathbf{Z}$, $0 < \alpha < 1$, and x is not a $(p+\alpha)$-spike for $R(X)$. Then there is a constant $\kappa > 0$ such that $d^p(x, y) \le \kappa |x-y|^\alpha$ for $y \in \Gamma$.*

Proof. Combine Theorem 1 and Theorem 2.

5. Next, we examine the structure of the set of τ-spikes. The case $\alpha = 0$ of the following lemma is due to Browder [1, p. 177].

Lemma. *Suppose μ is a Radon measure with no mass at x, $0 < b \in \mathbf{R}$, and $E^\alpha = \{y \in \mathbf{C}: |x-y|^{1+\alpha}\mu^{1+\alpha}(y) < b\}$. Then E^α has full area density at x, for $0 \le \alpha < 1$.*

Proof. For $r > 0$ let $\nu_r = \mathfrak{L}^2|(B(x, r)\setminus E^\alpha)$ (= area measure restricted to the complement of E^α). Then by the definition of E^α and Fubini's theorem,

$$\mathfrak{L}^2(B(x, r)E^\alpha)b = \|\nu_r\| b \le \int |x-y|^{1+\alpha}\mu^{1+\alpha}(y)\, d\nu_r(y) = \pi r^2 \int G_r(z)\, d|\mu|(z),$$

where

$$G_r(z) = \frac{1}{\pi r^2}\int \frac{|x-y|^{1+\alpha}}{|z-y|^{1+\alpha}}\, d\nu_r(y).$$

It is easy to see that $G_r(z)$ tends pointwise boundedly to zero on $\mathbf{C}\setminus\{x\}$, hence $\lim_{r\to 0}[\mathfrak{L}^2(B(x, r)\setminus E^\alpha)/\pi r^2] = 0$.

We note in passing that by applying the technique of [8, Lemma 2] a much stronger result may be obtained. Let C^β denote the *capacity of order* β: if $E \subset \mathbf{C}$, $0 < \beta \in \mathbf{R}$, then $C^\beta(E) = \sup\{|\nu(\mathbf{C})|: \nu$ is a Radon measure with support in E, $\nu^\beta \le 1\}$. Then, *if μ, x, b, E are as in the lemma, it follows that*

$$\sum_{n=1}^{+\infty} 2^{(1+\alpha)n} C^{1+\alpha}(A_n(x)\setminus E^\alpha) < +\infty.$$

In particular, for $\beta > 1 + \alpha$, the β-dimensional density at x of β-dimensional *Hausdorff content* M^β (cf. [8]), restricted to the complement of E^α, is zero.

Corollary 2. *Suppose x is not a peak point for $R(X)$, and $0 < \alpha < 1$. Then* *the set* $\{y \in X: y$ *is not an* α*-spike*$\}$ *has full area density at* x.

Proof. There is a representing measure μ for x with no mass at x [2, p. 54, 11.3]. Applying the lemma with $b = 1$ and $\alpha = 0, \alpha$ respectively we deduce that E^0, E^α, and hence $E^0 \cap E^\alpha$, have full area density at x. Set $\nu = (z-x)\mu$. Then, for $y \in E^0$, $\hat{\nu}(y) \neq 0$ and $\sigma = \hat{\nu}(y)^{-1}(z-y)^{-1}\nu = \hat{\nu}(y)^{-1}(z-x)(z-y)^{-1}\mu$ represents y on $R(X)$ [1, p. 176]. For $y \in E^0 \cap E^\alpha$, $\mu^{1+\alpha}(y) < +\infty$, hence

$$\sigma^{1+\alpha}(y) = |\hat{\nu}(y)|^{-1} \int \frac{|z-x|}{|z-y|^{1+\alpha}}\, d|\mu|(z) \le |\hat{\nu}(y)|^{-1} \cdot \operatorname{diam} X \cdot \mu^{1+\alpha}(y) < +\infty,$$

so $E^0 \cap E^\alpha$ consists entirely of non-α-spikes.

6. This enables us to strengthen Bishop's criterion [2, p. 54] for $R(X) = C(X)$ (= the space of all continuous functions on X). Bishop showed that *if* $\mathfrak{L}^2$ *almost all points of* X *are peak points for* $R(X)$, *then* $R(X) = C(X)$.

Theorem 3. *Let* $X \subset \mathbf{C}$ *be compact. Then* $R(X) = C(X)$ *if for* $\mathfrak{L}^2$ *almost every* $x \in X$ *there is* α, $0 < \alpha < 1$, *and* x *is an* α*-spike.*

Proof. By Corollary 2, every point of X is a peak point for $R(X)$, hence by Bishop's theorem, $R(X) = C(X)$.

A direct proof is also available: if ν is an annihilating measure for $R(X)$, then $\nu^{1+\alpha}(y) < +\infty$ for $\mathfrak{L}^2$ almost all y; if $\nu^{1+\alpha}(y) < +\infty$ and $\hat{\nu}(y) \neq 0$, then, constructing σ as in the proof of Corollary 2, we see that y is not an α-spike for $R(X)$, hence $\hat{\nu}(y) = 0$ for $\mathfrak{L}^2$ almost all y, hence $\nu = 0$ [2, p. 46, 8.2].

Corollary 3. $R(X) = C(X)$ *if for* $\mathfrak{L}^2$ *almost every* $x \in X$ *there is* α, $0 < \alpha < 1$, *with* $I_\alpha(X, x) = +\infty$.

Proof. Theorem 2 + Theorem 3.

Corollary 4. *Suppose for* $\mathfrak{L}^2$ *almost every* $x \in X$ *there exists* α, $0 < \alpha < 1$, *and*

$$\limsup_{r \to 0} \frac{\gamma(U(x, r) \setminus X)}{r^{1+\alpha}} > 0.$$

Then $R(X) = C(X)$.

This last fact was previously known; in fact it is known that α may be replaced by 1 [2, p. 207]. However, that result depends on the instability of analytic capacity, a very deep theorem. It is not possible to replace α by 1 in Corollary 3, for Wermer [10] has shown that there exist compact sets X such that $R(X)$ admits no bounded point derivations (hence, by Hallstrom, $I_1(X, x) = +\infty$ for all $x \in X$), yet $R(X) \neq C(X)$.

To prove Corollary 4 note that the argument of Case 2° of the proof of Theorem 2 shows that the lim sup condition implies x is an α-spike.

Fix $X \subset \mathbf{C}$, compact, and set $D^\tau = \{x \in X: I_\tau(X, x) < +\infty\}$. In [6] it was noted that D^0 never contains isolated points, while for $\tau \geq 1$, D^τ may consist of a single point. We are now in a position to complete the picture.

Corollary 5. *If* $0 < \tau < 1$, *then* D^τ *has full area density at each of its points.*

Proof. Each point x of D^τ belongs to D^0, hence is a nonpeak point, and by Corollary 2 the set of non-τ-spikes has full area density at x. By Theorem 2 every non-τ-spike is in D^τ.

When Gleason first introduced parts [3] he expressed the hope that there might be bounded (first order) point derivations at most points of a nontrivial part. While this hope was not borne out by the facts, the foregoing discussion shows that at most points of a part of $R(X)$ the functions in $R(X)$ just barely miss being differentiable, in the sense that $\mathfrak{L}^2$ almost all points of a part are not α-spikes for any α in $(0, 1)$.

We should mention that there are examples of points which are α-spikes but not peak points, so that the theory is not vacuous. For instance, consider a *Zalcman set*, a compact set X obtained by deleting from the closed unit disc a sequence of open balls B_n of radius r_n, with $B_n \subset A_n(0)$, $n = 1, 2, 3, \cdots$. Since $\gamma(A_n(0)\setminus X) = \gamma(B_n) = r_n$, Mel'nikov's theorem implies that 0 is a peak point for $R(X)$ if and only if $\sum_{n=1}^{+\infty} 2^n r_n = +\infty$. By Theorem 2, if $0 < \alpha < 1$, then 0 is an α-spike for $R(X)$ provided $\sum_{n=1}^{+\infty} 2^{(1+\alpha)n} r_n = +\infty$. Choose $\beta \in (1, 1+\alpha)$, $r_n = 2^{-(1+\beta)n}$. Then 0 is an α-spike but not a peak point. Incidentally, for a Zalcman set the converse to Theorem 2 is true: 0 is a τ-spike if *and only if* $I_\tau = +\infty$.

REFERENCES

1. A. Browder, *Introduction to function algebras*, Benjamin, New York, 1969. MR 39 #7431.

2. T. Gamelin, *Uniform algebras*, Prentice-Hall, Englewood Cliffs, N. J., 1969.

3. A. M. Gleason, *Function algebras*, Seminars on Analytic Function, vol. 2, Inst. for Advanced Study, Princeton, N. J., 1958.

4. A. Hallstrom, *On bounded point derivations and analytic capacity*, J. Functional Analysis 4 (1969), 153–165. MR 39 #4680.

5. M. S. Mel'nikov, *Estimates of the Cauchy integral along an analytic curve*, Mat. Sb. 71 (113) (1966), 503–514; English transl., Amer. Math. Soc. Transl. (2) 80 (1969), 243–255. MR 34 #6120.

6. A. O'Farrell, *An isolated bounded point derivation*, Proc. Amer. Math. Soc. 39 (1973), 559–562.

7. ———, *Equiconvergence of derivations*, Pacific J. Math. (to appear).

8. ———, *Density of parts of algebras on the plane*, Trans. Amer. Math. Soc. (to appear).

9. A. G. Vituškin, *Estimates of the Cauchy integral*, Mat. Sb. 71 (113) (1966), 515–534; English transl., Amer. Math. Soc. Transl. (2) 80 (1969), 257–278. MR 34 #6124.

10. J. Wermer, *Bounded point derivations on certain Banach algebras*, J. Functional Analysis 1 (1967), 28–36. MR 35 #5948.

11. D. R. Wilken, *Bounded point derivations and representing measures on* $R(X)$, Proc. Amer. Math. Soc. 24 (1970), 371–373. MR 40 #1781.

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