REMOVABLE SETS FOR CLASSES OF FUNCTIONS

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ABSTRACT. In a general context it is shown that a closed countable union of closed sets of removable singularities is again a set of removable singularities.

0. Introduction. Dolženko noted in [1], that *a closed countable union of closed sets of removable singularities for bounded analytic functions in the plane is also removable for such functions.* The sets of removable singularities for bounded analytic functions are the null sets of inner analytic capacity, γ, and the result is more striking since γ is not known to be subadditive. It turns out that the special character of the functions considered by Dolženko has little to do with the result. In this paper we formulate the notion of "set of removable singularities" (*F*-null set) in a general context, and we show that countable unions of such sets are again sets of the same type. This fact emerges as essentially an equivalent of the Baire Category Theorem. The argument is most naturally presented in the language of categories [3].

The "sheaf" case, which covers Dolženko's theorem, is dealt with in §1. Other cases are sketched in §3, while §§2 and 4 are devoted to examples.

The author is grateful to the referee for many helpful suggestions.

1. Removability in Baire spaces. Suppose X is a topological space, Y is a Hausdorff topological space, $\mathcal{O}$ is the category of open subsets of X, with inclusions as morphisms, and $C(U, Y)$, for $U \in \mathcal{O}$, is the class of continuous functions mapping U to Y. Let $\mathscr{F}_Y = \bigcup_{U \in \mathcal{O}} 2^{C(U,Y)}$. An element of $\mathscr{F}_Y$ is a set of continuous maps from some U to Y. $\mathscr{F}_Y$ forms a category, with restrictions as morphisms. Denote the restriction $C(U, Y) \to C(V, Y)$, where $V \subset U$, by $\rho(U, V)$. Let $F: \mathcal{O} \to \mathscr{F}_Y$ be a contravariant functor. Then for each $U \in \mathcal{O}$, $F(U)$ is some set of continuous maps from U to Y. Also $\rho(U, V)f \in F(V)$ whenever $V \subset U$ and $f \in F(U)$.

We say F is **local** if f belongs to $F(U)$ whenever $f \in C(U, Y)$, $\mathcal{U} \subset \mathcal{O}$ covers U, and $\rho(U, U \cap V)f \in F(U \cap V)$ for each $V \in \mathcal{U}$. We denote the

class of local F's by $CL(X, Y)$. The local F's are sometimes referred to as the sheaves of sets of Y-valued continuous functions. We say that a *closed* set $E \subset X$ is *F*-**null** if the restriction map $\rho(U, U \backslash E): F(U) \rightarrow F(U \backslash E)$ is bijective for each $U \in \mathcal{O}$. A closed subset of a closed F-null set is F-null. We say an arbitrary subset of X is *F*-**null** if each of its closed subsets is F-null. If $F(U)$ contains at least two elements whenever U is nonempty, we say F is **proper**. F cannot be proper unless Y contains at least two points. If F is proper, then any closed F-null set is nowhere dense. For if E is closed and $E^\circ \neq \varnothing$, then $F(E^\circ \backslash E) = F(\varnothing) \subset C(\varnothing, Y) = \{\varnothing\}$, hence card $F(E^\circ \backslash E^*) \leq 1 <$ card $F(E^\circ)$ hence $\rho(E^\circ, E^\circ \backslash E)$ is not injective on $F(E^\circ)$, so that E is not F-null. If we assume that every open subset of X contains a closed set with nonempty interior, then whenever F is proper, every F-null set is nowhere dense.

We say X is a **Baire space** if the Baire Category Theorem holds on X, i.e., if any countable union of closed nowhere dense sets has no interior. Examples of Baire spaces are provided by the locally compact Hausdorff spaces and the complete separable metric spaces. Every topological space is the disjoint union of an open subset of first category and a Baire subspace [4, p. 201]. Every open subset of a Baire space becomes a Baire space when endowed with the relative topology.

We say X **is of countable character** if, for every Hausdorff space Y and every proper $F \in CL(X, Y)$, any countable union of closed F-null sets is F-null.

THEOREM 1. *Every Baire space is of countable character.*

PROOF. Suppose X is a Baire space, Y is Hausdorff, $F \in CL(X, Y)$ is proper, $\{E_n\}_1^\infty$ is a sequence of closed F-null sets, and $E = \bigcup_{n=1}^{+\infty} E_n$. We wish to show that E is F-null. We may suppose E is closed. Clearly E is nowhere dense, since X is Baire.

Let us suppose E is not F-null, and seek a contradiction. There exists $U \in \mathcal{O}$ such that the map $\rho(U, U \backslash E): F(U) \rightarrow F(U \backslash E)$ is not bijective. The map cannot fail to be injective, since Y is Hausdorff and E is nowhere dense. Hence it is not surjective. So there is a function $f \in F(U \backslash E)$ which has no extension in $F(U)$. Form

$$\mathcal{B} = \{S \subset E : S \text{ is closed}, f \in \text{im } \rho(U \backslash S, U \backslash E) \mid F(U \backslash S)\}, \qquad T = \bigcap \mathcal{B}.$$

Then T is a closed subset of E. We claim that $T \in \mathcal{B}$.

The set $\{U \backslash S : S \in \mathcal{B}\}$ is an open covering of $U \backslash T$. The various extensions of f to $U \backslash S$'s agree on overlaps, since each S is nowhere dense and Y is Hausdorff; hence f has an extension $g \in C(U \backslash T, Y)$, and by localness $g \in F(U \backslash T)$. This proves the claim.

Now $T \cap U \neq \varnothing$, since otherwise $U \backslash T = U$. Also $T = \bigcup_{n=1}^{+\infty} (T \cap E_n)$. Since U is a Baire space there exist an integer n and an open set $V \subset U$ such that

$$\varnothing \neq T \cap V = E_n \cap T \cap V,$$

hence $V \backslash T = V \backslash (E_n \cap T)$.

Now $E_n \cap T$ is F-null, so the restriction $F(V) \to F(V \backslash T)$ is surjective. Since $g \in F(U \backslash T)$, $l = \rho(U \backslash T, V \backslash T)g \in F(V \backslash T)$, hence l has an extension $m \in F(V)$; since m and g agree on the overlap $V \backslash T$, localness ensures a common extension $n \in F(U \backslash (T \backslash V))$. This means that $T \backslash V \in \mathscr{B}$, hence $T = \bigcap \mathscr{B} \subset T \backslash V \subsetneqq T$. This is the desired contradiction, and the proof is complete.

2. Examples.

EXAMPLE 1.   Let X be locally connected, $Y = \{0, 1\}$ with the discrete topology. Let $F(U) = C(U, Y)$ for $U \in \mathcal{O}$. Then F is local and proper, and the elements of $F(U)$ are locally constant functions.

We say that a closed subset E of X **does not locally disconnect** X if

(1) E is nowhere dense, and

(2) $U \backslash E$ is connected for every connected open set $U \in \mathcal{O}$.

The closed F-null sets are precisely the closed sets which do not locally disconnect X, as is easily seen. So Theorem 1 yields a result about connectedness:

COROLLARY 1.   *Suppose E_n ($n = 1, 2, 3, \cdots$) are closed subsets of a locally connected Baire space X, and $E = \bigcup_{n=1}^{+\infty} E_n$ is closed. Suppose each E_n fails to locally disconnect X. Then E does not locally disconnect X.*

The corresponding global statement is false: a sphere disconnects $\mathbf{R}^3$, but is the union of hemispheres, which do not.

EXAMPLE 2. $X = C$, $Y = C$, $F(U) = \{f \in C(U, C): f$ is analytic on U and $\|f\|_U \leqq 1\}$. Here $\| \cdot \|_U$ denotes the uniform norm on U. F is proper and local, and the F-null sets are the sets of analytic capacity zero. Theorem 1 implies Dolženko's theorem as quoted in the introduction.

EXAMPLE 3. X is any analytic variety, $Y = C$,

$$F(U) = \{f \in C(U, C): f \text{ is analytic on } U \text{ and } \|f\|_U \leqq 1\}.$$

F is proper and local, and we obtain a generalization of Dolženko's theorem.

EXAMPLE 4.   Let X and Y be Banach spaces, X separable, and let $\Phi(x, p)$ be a differential operator:

$$\Phi: X \times Jet(X, Y) \to Y,$$

Φ of order ν. Let

$$F(U) = \{f \in C^\nu(U, Y) \mid \Phi(x, Jet_\nu f) = 0 \text{ for } x \in U, \|f\|_U \leq 1\}.$$

F is proper and local. The closed F-null sets are called the *sets of removable singularities for bounded solutions of* $\Phi = 0$.

COROLLARY 2. *A closed countable union of sets of removable singularities for the bounded solutions of* $\Phi = 0$ *is a set of the same type.*

EXAMPLE 5. The word "bounded" may be removed from Example 4 throughout. This is of interest in the case of overdetermined systems in high dimensions. Consider, for instance, analytic functions of two complex variables.

3. **Removability for classes of globally continuous functions.** Again, let X and Y be topological spaces, Y Hausdorff. Let $GCL(X, Y)$ denote the class of functors $G: \mathscr{O} \to 2^{C(X, Y)}$ such that

(1) $G(V) \subset G(U)$ wherever $U \subset V$,

(2) $G(U) = \bigcap \{G(V) \mid V \in \mathscr{U}\}$ whenever $\mathscr{U} \subset \mathscr{O}$ covers U,

(3) $G(\varnothing) = C(X, Y)$.

GCL stands for global, continuous, and local.

We say an element $G \in GCL(X, Y)$ is **proper** if $G(U) = C(X, Y)$ implies $U = \varnothing$.

A closed subset E of X is called G-**null** if $G(U \setminus E) = G(U)$ for all $U \in \mathscr{O}$. An arbitrary subset of X is called G-**null** if each of its closed subsets is G-null. Clearly this is consistent. We now state the analogue of Theorem 1 in this context.

THEOREM 2. *Let* X *be Baire,* Y *Hausdorff, and* $G \in GCL(X, Y)$ *be proper. Then a countable union of closed G-null sets is G-null.*

We omit the proof of Theorem 2. It parallels that of Theorem 1, but is simpler in two respects: all the $G(U)$ are subsets of the fixed set $C(X, Y)$, and the question of nonunique extensions does not arise.

The discussion of §1 may also be modified to deal with functors whose values are subsets of $L^p(X, Y, \mu)$, where X is a locally compact Hausdorff space, Y is a Banach space, μ is a positive Borel regular measure on X, $0 < p \leq +\infty$ and $L^p(X, Y, \mu)$ denotes the usual space of pth power summable μ-measurable functions with values in Y.

4. **Examples.**

EXAMPLE 6. Let $X = Y = C$, and

$$G(U) = A(U) = C(C, C) \cap \{f \mid f \text{ is analytic on } U\},$$

for every open set $U \subset C$. Then $G \in GCL(X, Y)$, and G is proper. The G-null sets are the so-called sets of continuous analytic capacity zero.

So we obtain another theorem of Dolženko, that a countable union of closed sets of zero continuous analytic capacity has zero continuous analytic capacity. More generally, X and Y may be taken to be Riemann surfaces.

EXAMPLE 7.   Let $X = Y = C$, $0 < \alpha \leq 1$, and

$$G(U) = A(U) \cap \mathrm{Lip}(C, \alpha),$$

whenever U is open in C. Then $G \in \mathrm{GCL}(X, Y)$, and G is proper. Again, countable unions of closed G-null sets are G-null sets. In case α is strictly less than 1, as Dolženko has shown [2], the G-null sets are precisely the sets of Hausdorff $(1+\alpha)$-dimensional measure zero, so the result may also be deduced from the sub-additivity of outer measures.

Again, X and Y may be replaced by metrized Riemann surfaces.

EXAMPLE 8.   Let X and Y be Banach spaces, X separable, and as in Example 4, let Φ be a (globally defined) differential operator of order ν,

$$\Phi : X \times Jet_\nu(X, Y) \to Y.$$

Let $G(U)$ denote the subset of $C(X, Y)$ consisting of those functions whose restrictions to U lie in $C^\nu(U, Y)$ and which satisfy $\Phi(x, Jet_\nu f) = 0$ on U. Then G is proper and local, so countable unions of closed sets of removable singularities for continuous solutions of $\Phi = 0$ are sets of the same type.

$G(U)$ may be modified by insisting in addition that $f \in \mathrm{Lip}(\alpha, X, Y)$, or that $f \in C^\tau(X, Y)$ for some positive integer τ. These G remain proper and local.

REFERENCES

1. E. P. Dolženko, *On the behaviour of a meromorphic function near its singularities*, Vestnik Moskov. Univ. Ser. I Mat. Meh. **1964**, no. 6, 3–6 (Russian) MR **32** #7749.

2. ———, *On removal of singularities of analytic functions*, Uspehi Mat. Nauk **18** (1963), no. 4, (112), 135–142; English transl., Amer. Math. Soc. Transl. (2) **97** (1971), 33–41. MR **27** #5898; **42** #5740.

3. H. Schubert, *Kategorien*. I, Heidelberger Taschenbücher, Bände 65, Springer-Verlag, Berlin and New York, 1970. MR **43** #311.

4. J. L. Kelley, *General topology*, Van Nostrand, Princeton, N.J., 1955. MR **16**, 1136.

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