

AN ISOLATED BOUNDED POINT DERIVATION¹

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ABSTRACT. For a compact subset X of the plane, $R(X)$ denotes the class of uniform limits on X of rational functions with poles off X . $R(X)$ is a function algebra on X . An example X is constructed such that $R(X)$ admits a bounded point derivation at exactly one point of X .

Let X be a compact subset of the plane C . We denote by $R(X)$ the uniform closure on X of the class $R_0(X)$ of rational functions with poles off X . Let x be a point of X . A *point derivation* on $R(X)$ at x is a linear functional D on $R(X)$ such that

$$D(fg) = f(x)Dg + g(x)Df,$$

for every pair f, g of elements of $R(X)$. D is called a *bounded point derivation* at x if it is continuous when $R(X)$ is given the topology induced by the uniform norm. It is easily seen that a bounded point derivation exists at x precisely when the map $D_0: R_0(X) \rightarrow C$, given by $D_0f = f'(x)$ is norm-continuous, and in this case there is exactly one bounded point derivation at x (up to constant multiples), namely, the extension of D_0 to $R(X)$.

If $x \in X^0$, then there is exactly one point derivation at x , and it is bounded. For boundary points there may be no derivations, no bounded derivations and infinitely many linearly independent unbounded derivations, or one (up to constant multiples) bounded derivation and infinitely many linearly independent unbounded derivations. These are the only three possibilities. The situation is neatly described in [1], [2]. Necessary and sufficient conditions for the existence of a bounded point derivation at a point $x \in X$ have been given in [3].

Browder's derivation theorem [2], and Browder's metric density theorem [1] together imply the following.

THEOREM 1. *The set of points of X at which point derivations of $R(X)$ exist contains no isolated points.*

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What makes this result nontrivial is the existence of sets X , with no interior, on which many derivations exist (cf. [1]).

We ask: Does Theorem 1 remain true if "point derivations" is replaced by "bounded point derivations"? The answer is no.

THEOREM 2. *There is a compact planar X such that $R(X)$ admits a bounded point derivation at 0, but has no other bounded point derivations.*

The object of this paper is to prove Theorem 2.

Let A_n denote the annulus $\{z \mid 1/2^{n+2} \leq |z| \leq 1/2^{n-1}\}$. By a Swiss Cheese we mean a compact set X lying in the closed unit disc D and containing the unit circle S obtained by deleting from D a countable union of discs D_i of radius r_i , the union of the D_i being dense in D . Swiss Cheeses have no interior, but provided $\sum r_i < +\infty$, it is well known [4] that $R(X) \neq C(X)$, the class of all continuous complex-valued functions on X .

Wermer [5] constructed Swiss Cheeses X such that $R(X)$ admits no bounded point derivations. Wermer's example has an additional property [5, p. 33, Line 13]:

LEMMA 1. *Let $t > 0$ be given. Then there is a Swiss Cheese X such that $R(X)$ admits no bounded point derivations, and $\sum r_i < t$.*

The next lemma is essentially due to Browder (cf. [5, Theorem 2]).

LEMMA 2. *Let X be a Swiss Cheese containing 0, and such that each D_i lies in some A_n . Let R_n denote $\sum r_i$, where the sum extends over those i such that D_i lies in A_n . Suppose $\sum_{n=1}^{+\infty} 4^n R_n < +\infty$. Then $R(X)$ admits a bounded point derivation at 0.*

PROOF. Choose f , rational with poles off X , bounded by 1 on X . We wish to show that $f'(0)$ is bounded independently of f .

Choose $K = D \cap \{z \mid |f(z)| \geq 2\|f\|_X\}$. Here $\|f\|_X$ denotes the uniform norm of f on X . K is covered by the D_i , so there is a positive integer N such that $K \subset \bigcup_{i=1}^N D_i$. There is another integer M such that $\bigcup_{i=1}^N D_i \subset \bigcup_{n=1}^M A_n$. By Cauchy's integral formula,

$$f'(0) = \frac{1}{2\pi i} \int_{\partial(\bigcup_{i=1}^N D_i)} \frac{f(z)}{z^2} dz.$$

Hence $|f'(0)| \leq (1/2\pi)(2\|f\|_X) \sum_{n=1}^M 4^{n+2} R_n \leq (16/\pi) \sum_{n=1}^{+\infty} 4^n R_n$. Q.E.D.

PROOF OF THEOREM 2. For each positive integer n choose (Lemma 1) a Swiss Cheese X_n such that $R(X_n)$ admits no bounded point derivations and $\sum r_i^{(n)} < 1/n^2 4^4$. Let \mathcal{D}_n denote the collection of "holes" in X_n which lie in A_n . Form $\mathcal{D} = \bigcup_{n=1}^{+\infty} \mathcal{D}_n$, $X = D \setminus (\bigcup \mathcal{D})$. Then X has the desired properties.

To see that X admits a bounded point derivation at 0, we use Lemma 2. An upper bound for R_n is

$$\sum_{m=n-2}^{n+2} r_i^{(m)} \leq 5 \cdot \frac{1}{(n-2)^2 4^{n-2}},$$

so clearly $\sum_{n=1}^{+\infty} 4^n R_n < +\infty$.

That $R(X)$ has no other bounded derivations follows from the observations:

- (1) Existence of bounded point derivations is a local question [3].
- (2) If $Z \subset Y$ and $R(Y)$ admits no bounded derivations, then $R(Z)$ admits no bounded derivations (trivial).
- (3) Locally, X is a subset of some X_n , by construction.

This concludes the proof.

REMARKS. 1. One may modify the construction of X to ensure that $R(X)$ admits a bounded point derivation of every order at 0, while having no other bounded point derivations. (One says that $R(X)$ admits a p th order bounded point derivation at x if the functional $f \rightarrow f^{(p)}(x)$ is bounded on $R_0(X)$.)

I am indebted to John Wermer for the above proof, which is considerably simpler than my original proof. I also wish to thank my advisor, Brian Cole, and Andrew Browder for their helpful comments.

2. Using the analytic capacity γ [6], this discussion may be recast in geometric language.

For $\lambda \in \mathbb{R}$, n a positive integer, $x \in \mathbb{C}$, and X compact, let

$$B_n(x) = \left\{ z \in \mathbb{C} \mid \frac{1}{2^{n+1}} \leq |z - x| \leq \frac{1}{2^n} \right\},$$

$$I_\lambda(X, x) = \sum_{m=1}^{\infty} 2^{m\lambda} \gamma(B_m(x) \setminus X).$$

We say that X is λ -dense at x if $I_\lambda(X, x) < +\infty$.

For $\lambda < 1$, λ -density is not an interesting notion: every X is everywhere λ -dense!

Melnikov's peak point theorem [6] states that x is a peak point for $R(X)$ if and only if X is not 1-dense at x . Combining this with Browder's metric density theorem we deduce that for every compact X the set $\{x \in \mathbb{C} \mid X \text{ is 1-dense at } x\}$ contains no isolated points. Theorem 2, together with Hallstrom's theorem [3], shows that this fails if 1 is replaced by 2. From Remark 1 and Hallstrom's theorem it follows that 1 may not be replaced by any $\lambda > 2$. We do not know whether or not 1 may be replaced by any λ with $1 < \lambda < 2$.

3. The p th order Gleason metric of $R(X)$ is defined for nonnegative integers p by

$$d^p(x, y) = \sup\{|f^{(p)}(x) - f^{(p)}(y)| \mid f \in R_0(X), \|f\|_X \leq 1\},$$

for $x, y \in X$. For $p=0$ this is the usual Gleason metric. For $p>0$, $d^p(x, y) < +\infty$ if and only if $R(X)$ admits a bounded p th order point derivation at both points. Theorem 2 shows that there can be no higher-order version of Browder's metric density theorem, for there are X for which the set $\{x \in X \mid R(X) \text{ admits a } p\text{th order bounded point derivation at } x\}$ contains a d^p -isolated point.

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