PROJECTS

PROFESSOR A. G. O'FARRELL

1. Current Problems

These are ones that I hope to clear up any day, now!

1.1. **Diophantine Approximation on a Cubic Curve.**

What is the Hausdorff dimension of

$$\left\{ (x, x^3) \in \mathbb{R}^2 : \left| x - \frac{p}{q} \right| + \left| x^3 - \frac{r}{q} \right| < \frac{1}{q^\tau}, \text{ infinitely-often} \right\}?$$

This is mysterious for $1 < \tau < 2$. It lies at the cutting edge of Diophantine approximation research.

You should understand that p, q and r are integers. (The question comes from Detta Dickinson.)

1.2. **Electrostatic Field of 3 Point Charges.**

Is it true that for generic points $x_i \in \mathbb{R}^3$ and generic charges $\mu_j \in \mathbb{R}$, the vector field

$$E(x) = \sum_{j=1}^{3} \frac{\mu_j (x - x_j)}{|x - x_j|^3}, \ (x \in \mathbb{R})$$

has at most 4 zeros? (J.C. Maxwell said yes, but his proof has a little gap. In 2004, Gabrielov, Novikov and Shapiro proved that there are at most 12 zeros.)

1.3. **Complex Polynomial Approximation.**

If f is a compex-valued continuous function on a compact $X \subset \mathbb{C}$ having connected complement, and if f is holomorphic and has no zero on the interior of X, is it always possible to approximate f , uniformly on X, by (analytic) polynomials having no zero on X? (From Johan Andersson, via Larry Zalcman.)

Date: September 3, 2011
This is a draft.

1.4. **Reversible Biholomorphic Maps.** Which are the reversible elements in the group of biholomorphic map germs at the origin in $\mathbb{C}^2$?

An element of a group G is reversible in G if it is conjugate to its inverse.

Dmitri Zaitsev and I have recently made some progress on this. We identified the *generic* reversibles. It remains to sort it out for some special classes of reversibles. For instance, which maps F having linear part $(z_1 + z_2, z_2)$ are reversible?

This problem is an example of a reversibility problem, the one nearest the top of my list at present. But if you have a favourite group, then the question may be open for that, too.

1.5. **Biholomorphic Germs in One Variable.** Is every biholomorphic germ with multiplier ± 1 the product of four involutions?

The multiplier of $f(z) = a_1 z + a_2 z + \cdots$ is the number a_1.

We know that this factorization can be done using formal germs, and recently, with Dmitri Zaitsev, I extended the formal factorization theorem to germs in arbitrary dimensions.

2. Problems on the Back Burner

These are things that have bothered me for decades:

2.1. **Harmonic Approximation.** Given a compact set $K \subset \mathbb{R}^d$ and a continuous function $f : K \to \mathbb{R}$, the problem is to decide when f is the uniform limit on K of a sequence of functions $ffng$, each harmonic near K. There are known solutions to this problem in terms of Brownian motion, but this is not good enough. There is a conjectured solution in terms of capacities.

In June, 2011, I received a preprint from Maxim Mazaloff containing a solution. He proved that the capacitary condition works.

2.2. **The f^2 Problem.** Suppose $K \subset \mathbb{C}$ is compact, and let $R(K)$ denote the algebra of all uniform limits on K of sequences of rational functions with poles off K. Suppose f is continuous on K and $f^2 \in R(K)$. Must f belong to $R(K)$? This relates to generalisation of Rado's theorem, and there are some partial results.

2.3. **Derivations on $R(K)$.** Another problem about $R(K)$ is whether $R(K) = C(K)$ as soon as there is no continuous derivation from $R(K)$ into a Banach $R(K)$-module.

2.4. **The graph of a direction-reversing homeomorphism of C.** Let $f : \mathbb{C} \to \mathbb{C}$ be a homeomorphism of degree -1. Must the graph of f be polynomially-convex? There are results for smooth f . There are many other problems about polynomial convexity.

2.5. **Nachbin's problem.** Describe the closure of a subalgebra of the topological algebra of all smooth real-valued functions on a smooth manifold.

3. Suggested Directions

These are not specific problems, but suggestions for research programs.

3.1. **Reversibility.**

3.2. **Negative Lipschitz Spaces.**

3.3. **Pervasive Spaces.**