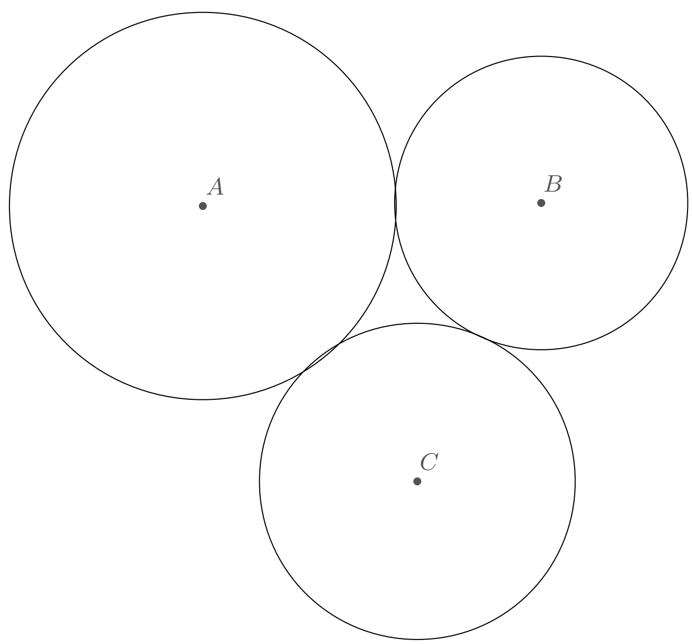
PROBLEM CHALLENGE 2012

SOLUTIONS TO PROF. O'FARRELL
DEPT OF MATHS AND STATS

(1) A stick is cut through at random in two places. What is the probability that the three peices can be arranged into a triangle?
(2) Show that if $m > 2$ and n are positive integers, then $2^m - 1$ *does not* divide $2^n + 1$.
(3) Given that each positive integer is the sum of four squares (of nonengative integers), show that each positive multiple of six is the sum of 48 fourth powers.
(4) Here are three mutually-tangent circles, with centres A, B and C:



Is it always possible to find three such circles, no matter where the points A, B and C are positioned? (Justify your answer!)
(5) Find, if possible, integers x and y such that

$$212x^2 + 100xy + 273y^2 = 1.$$

(6) Evaluate the $n \times n$ determinant

$$A_n = \begin{pmatrix} 2 & \mu & 0 & 0 & \cdots & 0 \\ \mu & 2 & \mu & 0 & \cdots & 0 \\ 0 & \mu & 2 & \mu & \cdots & 0 \\ . & . & . & . & \cdots & . \\ . & . & . & . & \cdots & . \\ 0 & 0 & 0 & 0 & \cdots & \mu \end{pmatrix}.$$

(7) If a_i $(i = 1, \ldots 4)$ are the roots of

$$x^4 + px^3 + rx + s = 0,$$

show that the monic polynomial equation with roots

$$a_1a_2 + a_3a_4, a_1a_3 + a_2a_4, a_1a_4 + a_2a_3$$

is

$$t^3 + (pr - 4s)t - (p^2s + r^2) = 0.$$

Hence solve

$$x^4 - x^3 + 2x - 1 = 0.$$

(8) Prove

$$\tan(\alpha-\beta)+\tan(\beta-\gamma)+\tan(\gamma-\alpha) = \tan(\alpha-\beta)\cdot\tan(\beta-\gamma)\cdot\tan(\gamma-\alpha)$$

for all real α,β,γ.

(9) Evaluate $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)}$.

(10) (Unsolved last year) Find a differentiable function $f : \mathbb{R}^2 \to \mathbb{R}$ that has a strict local maximum at $(0, 0)$ when restricted to each straight line through $(0, 0)$, but does not have a local minimum at $(0, 0)$.

On the Problem Board, a sticker beside a problem indicates that at least one correct solution has been submitted. Please do not write solutions on the board. This spoils the fun for others. Just drop by my office, or leave a note, or send an email.

E-mail address: admin@maths.nuim.ie