REVERSIBILITY IN THE GROUP OF HOMEOMORPHISMS OF THE CIRCLE

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ABSTRACT

The group of orientation preserving homeomorphisms of the circle is simple, and, because there are non-trivial involutions in this group, it must be generated by its involutions. We show that, in this group of homeomorphisms, each element can be expressed as a product of three involutions. We also characterise those elements of the group that can be expressed as a composite of two involutions, and perform a similar characterisation in the full group of homeomorphisms of the circle.

1. Introduction

We describe an element g of a group G as *reversible* in G if it is conjugate in G to its own inverse. We say that g is *strongly reversible* in G if there exists an involution τ in G such that $\tau g \tau = g^{-1}$. This language has developed from the theory of finite groups, where the terms *real* and *strongly real* replace *reversible* and *strongly reversible*. (The word *real* is used because an element g of a finite group G is reversible if and only if each irreducible character of G takes a real value when applied to g.) Notice that g is strongly reversible if and only if it can be expressed as a composite of two involutions.

Let $\mathbb{S}$ denote the unit circle in $\mathbb{R}^2$ centred on the origin. Denote by $\mathcal{H}$ the group of homeomorphisms of $\mathbb{S}$. There is a simple subgroup $\mathcal{H}^+$ of $\mathcal{H}$ consisting of orientation preserving homeomorphisms. The subgroup $\mathcal{H}^+$ has a single distinct coset $\mathcal{H}^-$ in $\mathcal{H}$ which consists of orientation reversing homeomorphisms. Since $\mathcal{H}^+$ is simple, and there are non-trivial involutions in $\mathcal{H}^+$, it must be generated by its involutions. In this paper we prove that each element of $\mathcal{H}^+$ can be expressed as a product of three involutions. Also, we classify the the strongly reversible maps in both groups $\mathcal{H}^+$ and $\mathcal{H}$.

A classification of the reversible maps in $\mathcal{H}^+$ and $\mathcal{H}$ can be extracted from a conjugacy classification in these two groups. We describe the conjugacy classes of $\mathcal{H}^+$ and $\mathcal{H}$ in §2, and comment briefly on reversibility. Note that the strongly reversible maps in the group of homeomorphisms of the real line were determined by Jarczyk and Young; see [**4**, **6**, **7**].

For points a and b in $\mathbb{S}$, we write (a, b) to indicate the open anticlockwise interval from a to b in $\mathbb{S}$. Let $[a, b]$ denote the closure of (a, b). For a proper open interval I in $\mathbb{S}$, we say that $u < v$ *in* I if $(u, v) \subset I$. To classify the strongly reversible maps in $\mathcal{H}^+$ we need the notion of the *signature* of an orientation preserving homeomorphism which has a fixed point. If f is such a homeomorphism, then each point x in $\mathbb{S}$ is either a fixed point of f or else it lies in an open interval component I in the complement of the fixed point set of f. The signature Δ_f of f is the function from $\mathbb{S}$ to $\{-1, 0, 1\}$ given by the equation

$$\Delta_f(x) = \begin{cases} 1 & \text{if } x < f(x) \text{ in } I, \\ 0 & \text{if } f(x) = x, \\ -1 & \text{if } f(x) < x \text{ in } I. \end{cases}$$

The strongly reversible maps in $\mathcal{H}^+$ are classified according to the next theorem, proven in §3.

THEOREM 1.1. *An element f of $\mathcal{H}^+$ is strongly reversible if and only if either it is an involution or else it has a fixed point and there is an orientation preserving homeomorphism h of rotation number $\frac{1}{2}$ such that $\Delta_f = -\Delta_f \circ h$.*

Elements in $\mathcal{H}^+$ that cannot be expressed as a composite of two involutions (elements that are not strongly reversible) can nevertheless be expressed as a composite of three involutions. The next theorem is proven in §4.

THEOREM 1.2. *Each member of $\mathcal{H}^+$ can be expressed as a composite of three orientation preserving involutions.*

A simple corollary of Theorem 1.2 is that $\mathcal{H}^+$ is uniformly perfect, meaning that there is a positive integer N such that each element of $\mathcal{H}^+$ can be expressed as a composite of N or fewer commutators. Since it is easy to express the rotation by π as a commutator, and each involution in $\mathcal{H}^+$ is conjugate to the rotation by π, it follows from Theorem 1.2 that $\mathcal{H}^+$ is uniformly perfect with $N = 3$. In fact, Eisenbud, Hirsch, and Neumann [1] proved that $\mathcal{H}^+$ is uniformly perfect with $N = 1$.

We move on to describe the strongly reversible elements in the larger group $\mathcal{H}$. There are orientation preserving homeomorphisms of $\mathbb{S}$ that are strongly reversible in $\mathcal{H}$, but not strongly reversible in $\mathcal{H}^+$; for example, rotations by angles other than 0 or π. Before we state our theorem on strong reversibility in $\mathcal{H}$, we introduce some notation which is explained in more detail in §2. For a homeomorphism f, the *degree* of f, denoted $\deg(f)$, is equal to 1 if f preserves orientation, and -1 if f reverses orientation. Let $\rho(f)$ denote the rotation number of an orientation preserving homeomorphism f. The rotation number is an element of $[0, 1)$. If $\rho(f) = 0$ then f has a fixed point. If $\rho(f)$ is rational then f has a periodic point, in which case the minimal period of f (the smallest positive integer n such that f^n has fixed points) is denoted by n_f. If $\rho(f)$ is irrational then we denote the minimal set of f, that is, the smallest non-trivial f invariant compact subset of $\mathbb{S}$, by K_f. This set is either a perfect and nowhere dense subset of $\mathbb{S}$ (a Cantor set) or else equal to $\mathbb{S}$. In the former case we define I_f to be the set of inaccessible points of K_f, and in the latter case we define I_f to be $\mathbb{S}$. There is a continuous surjective map $w_f : \mathbb{S} \to \mathbb{S}$ of degree 1 with the properties: (i) w_f maps I_f homeomorphically onto $w_f(I_f)$; (ii) w_f maps each closed interval component in the complement of I_f to a point; and (iii) $w_f f = R_\theta w_f$, where R_θ is the anticlockwise rotation by $\theta = 2\pi\rho(f)$. The next theorem is proven in §5.

THEOREM 1.3. *Let f be an orientation preserving member of $\mathcal{H}$. Either*

(i) *$\rho(f) = 0$, in which case f is strongly reversible if and only if there is a homeomorphism h such that $\Delta_f = -\deg(h) \cdot \Delta_f \circ h$, and either h preserves orientation and has rotation number $\frac{1}{2}$ or else reverses orientation;*

(ii) *$\rho(f)$ is non-zero and rational, in which case f is strongly reversible if and only if f^{n_f} is strongly reversible by an orientation reversing involution;*

(iii) *$\rho(f)$ is irrational, in which case f is strongly reversible if and only if $w_f(I_f)$ has a reflectional symmetry.*

Using Theorem 1.3 we also show that an orientation preserving circle homeomorphism is strongly reversible by an orientation *reversing* involution if and only if it is reversible by an orientation *reversing* homeomorphism.

It remains to state a result on strong reversibility of orientation reversing homeomorphisms. Each orientation reversing homeomorphism has exactly two fixed points. The next theorem is proven in §6.

THEOREM 1.4. *An orientation reversing homeomorphism f is strongly reversible if and only if there is an orientation reversing homeomorphism s that interchanges the pair of fixed points of f and satisfies $sf^2s^{-1} = f^{-2}$.*

Fine and Schweigert proved in [**2**, Theorem 25] that each member of $\mathcal{H}$ can be expressed as a composite of three involutions (this follows quickly from Theorems 1.2 and 1.3), and that there are elements that cannot be expressed as a composite of two involutions.

We now describe the structure of this paper. Section 2 contains unoriginal material; it consists of a brief review of a conjugacy classification in $\mathcal{H}^+$ and $\mathcal{H}$. All subsequent sections contain new results. Sections 3 and 4 are about $\mathcal{H}^+$ and Sections 5 and 6 are about $\mathcal{H}$.

2. Conjugacy classification

Two conjugacy invariants which can be used to determine the conjugacy classes in $\mathcal{H}$ are the rotation number and signature, introduced in §1. We describe here only those properties of these two quantities that we use. For more information on rotation numbers, see [**3**]; for more information on signatures, see [**2**].

For an orientation preserving homeomorphism f, choose any point x in $\mathbb{S}$, and let θ_n be the angle in $[0, 2\pi)$ measured anticlockwise between $f^{n-1}(x)$ and $f^n(x)$. The *rotation number* of f, denoted $\rho(f)$, is the unique number in $[0, 1)$ such that the expression

$$(\theta_1 + \cdots + \theta_n) - 2\pi n \rho(f)$$

is bounded for all n. The quantity $\rho(f)$ is independent of x. The rotation number is invariant under conjugation in $\mathcal{H}^+$, and $\rho(f^n) = n\rho(f) \pmod 1$, for each integer n.

A straightforward consequence of the definition of $\rho(f)$ is that $\rho(f) = 0$ if and only if f has a fixed point. In this case, $\mathbb{S}$ can be partitioned into a closed set $\mathrm{fix}(f)$, consisting of fixed points of f, and a countable collection of open intervals on each of which f is free of fixed points. Now suppose that $\rho(f) = p/q$, where p and q are coprime positive integers. Then f has periodic points, that is, there is a positive integer n for which f^n has fixed points. The smallest such n, denoted n_f, is the *minimal period* of f, and is equal to q.

The remaining possibility is that $\rho(f)$ is irrational. In this case we define K_f to be the unique minimal set in the poset consisting of f invariant compact subsets of $\mathbb{S}$ ordered by inclusion. We describe K_f as the *minimal set* of f. Either $K_f = \mathbb{S}$ or else K_f is a perfect subset of $\mathbb{S}$ with empty interior—a Cantor set. In the latter case there is a sequence of open intervals (a_i, b_i), for $i = 1, 2, \ldots$, such that $[a_i, b_i] \cap [a_j, b_j] = \emptyset$ when $i \neq j$, and K_f is the complement of $\bigcup_{i=1}^{\infty}(a_i, b_i)$. The set I_f of *inaccessible points* of K_f is the complement of $\bigcup_{i=1}^{\infty}[a_i, b_i]$. If $K_f = \mathbb{S}$ then we define $I_f = \mathbb{S}$. There is a continuous surjective map w_f of $\mathbb{S}$ of degree 1 such that $w_f f = R_\theta w_f$, where $\theta = 2\pi\rho(f)$. The map w_f is chosen such that it is a homeomorphism when restricted to I_f, and it maps each interval $[a_i, b_i]$ to a single point. This map w_f is unique up to post composition by rotations.

Now suppose that f is an orientation preserving homeomorphism that has a fixed point. The signature of f was defined in §1. The signature Δ_f takes the value 0 on $\mathrm{fix}(f)$, and elsewhere it takes either the value -1 or the value 1. Useful properties of the signature are encapsulated in the next elementary lemma.

LEMMA 2.1. *If f is a member of $\mathcal{H}^+$ with a fixed point, and h is a member of $\mathcal{H}$, then*
(i) $\Delta_{hfh^{-1}} = \deg(h) \cdot \Delta_f \circ h^{-1}$,

(ii) $\Delta_{f^{-1}} = -\Delta_f$.

We are now in a position to state criteria which determine whether two circle homeomorphisms are conjugate. The results are stated in such a way that one can deduce from them when two orientation preserving circle homeomorphisms are conjugate in each of the groups $\mathcal{H}^+$ and $\mathcal{H}$. The result on irrational rotation numbers follows from [**5**, Theorem 2.3]. The result on orientation preserving homeomorphisms with fixed points is similar to [**2**, Theorem 10]. The other two theorems are well-known; they can both be proven directly. Recall that the degree of a circle homeomorphism is 1 if the map preserves orientation, and -1 if it reverses orientation.

THEOREM 2.2. *Two orientation preserving circle homeomorphisms f and g, each of which has a fixed point, are conjugate by a homeomorphism of degree ϵ if and only if there is a homeomorphism h of degree ϵ such that $\Delta_g = \epsilon \Delta_f \circ h$.*

Note in particular that a map g in $\mathcal{H}^+$ with fixed points is conjugate in $\mathcal{H}^+$ to all of its positive powers.

THEOREM 2.3. *Two orientation preserving circle homeomorphisms f and g, both of which have the same non-zero rational rotation number, are conjugate by a homeomorphism of degree ϵ if and only if f^{n_f} is conjugate to g^{n_g} by a homeomorphism of degree ϵ.*

Since f and g have the same non-zero rational rotation number, the integers n_f and n_g in Theorem 2.3 (the minimal periods of f and g) are equal.

Recall that an orthogonal map of the circle of degree 1 is a rotation, and an orthogonal map of the circle of degree -1 is a reflection in a line through the origin.

THEOREM 2.4. *Two orientation preserving circle homeomorphisms f and g, both of which have the same irrational rotation number, are conjugate by a homeomorphism of degree ϵ if and only if there is an orthogonal map of degree ϵ that maps $w_f(I_f)$ to $w_g(I_g)$.*

It remains to consider conjugacy between orientation reversing maps.

THEOREM 2.5. *Two orientation reversing circle homeomorphisms f and g are conjugate in $\mathcal{H}$ if and only if f^2 and g^2 are conjugate in $\mathcal{H}$ by a homeomorphism that maps the pair of fixed points of f to the pair of fixed points of g.*

It follows from Theorems 2.3 and 2.5 that all non-trivial involutions in $\mathcal{H}^+$ are conjugate, and all orientation reversing involutions in $\mathcal{H}$ are conjugate. (These statements can easily be seen directly.)

We briefly remark on the reversible elements in $\mathcal{H}^+$ and $\mathcal{H}$. Suppose that f and h are members of $\mathcal{H}^+$ such that $hfh^{-1} = f^{-1}$. Then

$$\rho(f^{-1}) = \rho(hfh^{-1}) = \rho(f) \pmod 1.$$

But $\rho(f^{-1}) = -\rho(f) \pmod 1$, hence $\rho(f)$ is equal to either 0 or $\frac{1}{2}$. On the other hand, if h reverses orientation and still $hfh^{-1} = f^{-1}$, then

$$\rho(hfh^{-1}) = -\rho(f) \pmod 1,$$

so, in this case, the rotation number tells us nothing about reversibility. Notice that if we compare Theorems 2.2, 2.3, and 2.4 with Theorem 1.3, we see that that an orientation preserving map f is reversible by an orientation *reversing* map if and only if f is strongly reversible by an orientation *reversing* involution. There are, however, orientation preserving homeomorphisms

that are not strongly reversible in $\mathcal{H}$, but are nevertheless reversible by orientation *preserving* maps; one example is given at the end of §5.

3. Proof of Theorem 1.1

The following two theorems deal with strong reversibility in $\mathcal{H}^+$ for the two rotation numbers 0 and $\frac{1}{2}$ separately.

A result similar to Theorem 3.2, below, has been proven by Jarzcyk [4] and Young [7] for homeomorphisms of the real line. We use an elementary lemma in the proof of Theorem 3.2.

LEMMA 3.1. *If f and g are orientation preserving homeomorphisms with fixed points such that $\Delta_f = \Delta_g$, then there is an orientation preserving homeomorphism k that fixes each of the fixed points of f and g such that $kfk^{-1} = g$.*

Proof. We define a homeomorphism k as follows. On each fixed point x of f and g, define $k(x) = x$. On each open interval component (a, b) of $\mathbb{S} \setminus \text{fix}(f)$, the signature function takes either the value 1 for both functions f and g, or else it takes the value -1 for both functions. If we choose an orientation preserving homeomophism $s : (a, b) \to \mathbb{R}$, then sfs^{-1} and sgs^{-1} are fixed point free homeomorphisms of $\mathbb{R}$, and either both maps are conjugate to $x \mapsto x + 1$, or both maps are conjugate to $x \mapsto x - 1$. Thus in either case we can choose an orientation preserving homeomorphism k_0 of (a, b) such that $k_0 f k_0^{-1}(x) = g(x)$ for x in (a, b). We then define $k(x) = k_0(x)$ for x in (a, b). We have constructed the required function k. Of course, the existence of a conjugation between f and g follows from Theorem 2.2, but we also need the property of k that it fixes each element of $\text{fix}(f)$. $\square$

We remark that the degree 1 case of Theorem 2.2 follows quickly from Lemmas 2.1 and 3.1.

THEOREM 3.2. *An element f of $\mathcal{H}^+$ with a fixed point is strongly reversible in $\mathcal{H}^+$ if and only if there is a homeomorphism h in $\mathcal{H}^+$ with rotation number $\frac{1}{2}$ such that $\Delta_f = -\Delta_f \circ h$.*

Proof. If $\sigma f \sigma = f^{-1}$ for an orientation preserving involution σ, then $\Delta_f = -\Delta_f \circ \sigma$, by Lemma 2.1. Conversely, suppose that there is a homeomorphism h in $\mathcal{H}^+$ with rotation number $\frac{1}{2}$ such that $\Delta_f = -\Delta_f \circ h$. By Lemma 2.1, $\Delta_{h^{-1}fh} = \Delta_{f^{-1}}$. Using Lemma 3.1 we can construct a map k in $\mathcal{H}^+$ that fixes each fixed point of f, and satisfies $k^{-1}h^{-1}fhk = f^{-1}$.

Now choose a fixed point q of f. The equation $\Delta_f = -\Delta_f \circ h$ tells us that all iterates of q under h also lie in the closed set $\text{fix}(f)$. Since h^2 has fixed points, the sequence $h^2(q), h^4(q), h^6(q), \dots$ converges; its limit is another point p in $\text{fix}(f)$, and $h^2(p) = p$. The points p and $h(p)$ are distinct elements of $\text{fix}(f)$ because h has no fixed points. Define

$$\mu(x) = \begin{cases} hk(x) & \text{if } x \in [p, h(p)], \\ k^{-1}h^{-1}(x) & \text{if } x \in [h(p), p]. \end{cases}$$

One can check that μ is an involution in $\mathcal{H}^+$ and $\mu f \mu = f^{-1}$. $\square$

We move on to orientation preserving homeomorphisms with rotation number $\frac{1}{2}$.

THEOREM 3.3. *An element of $\mathcal{H}^+$ with rotation number $\frac{1}{2}$ is strongly reversible in $\mathcal{H}^+$ if and only if it is an involution.*

Proof. All involutions are strongly reversible by the identity map. Conversely, let f be a homeomorphism with rotation number $\frac{1}{2}$, and let σ be an involution in $\mathcal{H}^+$ such that

$\sigma f \sigma = f^{-1}$. Choose an element x of fix(f^2). Then $f(x)$ is also an element of fix(f^2), and by interchanging x and $f(x)$ if necessary, we may assume that $\sigma(x) \in (x, f(x)]$. Suppose that $\sigma(x) \neq f(x)$. Since σ maps $(\sigma(x), x)$ onto $(x, \sigma(x))$, we have that $\sigma f(x) \in (x, \sigma(x))$. Likewise, f maps $(x, f(x))$ onto $(f(x), x)$, therefore $f\sigma(x) \in (f(x), x)$. However, $f\sigma(x) = \sigma f^{-1}(x) = \sigma f(x)$, and yet $(x, \sigma(x)) \cap (f(x), x) = \emptyset$. This is a contradiction, therefore $\sigma(x) = f(x)$. This means that σf, which is an orientation preserving involution, fixes x; hence it is the identity map. Therefore $f = \sigma$, as required. $\square$

In contrast to Theorem 3.3 there are reversible homeomorphisms with rotation number $\frac{1}{2}$ that are not strongly reversible. An example is given at the end of §5.

We have all the ingredients for a proof of Theorem 1.1.

Proof of Theorem 1.1.   If f is a strongly reversible member of $\mathcal{H}^+$ then it is reversible, so it must have rotation number equal to either 0 or $\frac{1}{2}$. If it has rotation number 0 then there is an orientation preserving homeomorphism h with rotation number $\frac{1}{2}$ such that $\Delta_f = -\Delta_f \circ h$, by Theorem 3.2. If f has rotation number $\frac{1}{2}$ then it is an involution, by Theorem 3.3. The converse implication follows immediately from Theorem 3.2. $\square$

4.   *Proof of Theorem 1.2*

Proof of Theorem 1.2.   Choose an element f of $\mathcal{H}^+$ that is not an involution. There exists a point x in $\mathbb{S}$ such that x, $f(x)$, and $f^2(x)$ are three distinct points. By replacing f with f^{-1} if necessary we can assume that x, $f(x)$, and $f^2(x)$ occur in that order anticlockwise around $\mathbb{S}$. Notice that $f^{-1}(x)$ lies in $(f(x), x)$. Select a point y in $(x, f(x))$ that is sufficiently close to $f(x)$ that $f^{-1}(y) \in (f^2(x), x)$. Choose an orientation preserving homeomorphism g from $[x, f(x)]$ to $[f(x), x]$ such that $g(y) = f(y)$, $g(t) < \min(f(t), f^{-1}(t))$ in (x, y), and $f(t) < g(t) < f^{-1}(t)$ in $(y, f(x))$. A graph of such a function is shown in Figure 1. Define an involution σ in $\mathcal{H}^+$ by

FIGURE 1.

the equation

$$\sigma(t) = \begin{cases} g(t) & \text{if } t \in [x, f(x)], \\ g^{-1}(t) & \text{if } t \in [f(x), x]. \end{cases}$$

Let us determine the fixed points of σf. For a point t in $[x, f(x)]$ we have that $\sigma f(t) = t$ if and only if $f(t) = g(t)$. This means that either $t = x$ or $t = y$. For a point t in $(f(x), x)$ we have that $\sigma f(t) = t$ if and only if $\sigma f g(w) = g(w)$, where $w = g^{-1}(t)$ is a point in $(x, f(x))$. Therefore $g(w) = f^{-1}(w)$. This equation has no solutions in $(x, f(x))$, hence σf has no fixed points in $(f(x), x)$. The direction of flow of σf on the complement of $\{x, y\}$ can also easily be deduced from the construction of g, and we find that

$$\Delta_{\sigma f}(t) = \begin{cases} 0 & \text{if } t = x, y, \\ 1 & \text{if } t \in (x, y), \\ -1 & \text{if } t \in (y, x). \end{cases}$$

By Theorem 3.2, σf is expressible as a composite of two involutions in $\mathcal{H}^+$. Therefore f is expressible as a composite of three involutions in $\mathcal{H}^+$. $\square$

5. Proof of Theorem 1.3

For the remainder of this document we work in the full group of homeomorphisms of the circle. In §2 we showed that all reversible maps in $\mathcal{H}^+$ have rotation number either 0 or $\frac{1}{2}$. This is not the case in $\mathcal{H}$ because if h is an orientation reversing map, and f an orientation preserving map, then $\rho(hfh^{-1}) = -\rho(f)$. Since also $\rho(f^{-1}) = -\rho(f)$, the rotation number tells us nothing about reversibility by orientation reversing homeomorphisms. In fact, since all rotations are strongly reversible in $\mathcal{H}$ by reflections, there are strongly reversible maps in $\mathcal{H}$ with any given rotation number.

We divide our analysis of strongly reversible maps in $\mathcal{H}$ between three cases corresponding to when the rotation number is 0, rational, or irrational. The first case is Theorem 1.3 (i). We need a preliminary lemma.

LEMMA 5.1.   *If f is a fixed point free homeomorphism of an open proper arc A in $\mathbb{S}$, then f is conjugate to f^{-1} on A by an orientation reversing involution of A.*

Proof.   The situation is topologically equivalent to the situation when f is a fixed point free homeomorphism of the real line. Such maps f are conjugate to non-trivial translations, and translations are reversible by the orientation reversing involution $x \mapsto -x$.   □

Proof of Theorem 1.3 (i).   If $\tau f \tau = f^{-1}$ for an involution τ, then $\Delta_f = -\deg(\tau) \cdot \Delta_f \circ \tau$, by Lemma 2.1. For the converse, we are given a homeomorphism h that satisfies $\Delta_f = -\deg(h) \cdot \Delta_f \circ h$. Either h preserves orientation and satisfies $\rho(h) = \frac{1}{2}$, in which case the result follows from Theorem 3.2, or else h reverses orientation.

In the latter case, by Lemmas 2.1 and 3.1 there is an orientation preserving homeomorphism k that fixes the fixed points of f, and satisfies $k^{-1}h^{-1}fhk = f^{-1}$. Define $s = k^{-1}h^{-1}$. As s is an orientation reversing homeomorphism, it has exactly two fixed points p and q. Let a denote the point in $\mathrm{fix}(f)$ that is clockwise from p, and closest to p. Possibly $a = p$. Define b to be the point in $\mathrm{fix}(f)$ that is anticlockwise from p and closest to p. Let $I = (a, b)$. Possibly $I = \emptyset$. Similarly we define an interval J about q. Since a point z in (p, q) is a fixed point of f if and only if the point $s(z)$ in (q, p) is a fixed point of f, we see that I and J do not intersect. Now, f fixes I so we can, by Lemma 5.1, choose an orientation reversing involution τ_I of I such that $\tau_I f \tau_I(x) = f^{-1}(x)$ for x in I. Similarly we define τ_J. From the equation $sfs^{-1} = f^{-1}$ and the definition of the intervals I and J we deduce that s fixes I and J. Hence we can define

$$\mu(x) = \begin{cases} \tau_I(x) & \text{if } x \in I, \\ \tau_J(x) & \text{if } x \in J, \\ s(x) & \text{if } x \in [p, q] \setminus (I \cup J), \\ s^{-1}(x) & \text{if } x \in [q, p] \setminus (I \cup J). \end{cases} \tag{5.1}$$

One can check that μ is an orientation reversing involution that satisfies $\mu f \mu = f^{-1}$.   □

To prove Theorem 1.3 (ii) we use a lemma that enables us to deal with rational rotation numbers of the form $1/n$, rather than m/n. Recall that n_f denotes the minimal period of f.

LEMMA 5.2.   *Let f be an element of $\mathcal{H}^+$ with a periodic point. If f^d is strongly reversible for an integer d in $\{1, 2, \ldots, n_f - 1\}$ that is coprime to n_f, then f is strongly reversible.*

Proof.   There exist integers u and t such that $dt = 1 + un_f$. Let q be the positive integer between 0 and n_f, and coprime to n_f, such that $\rho(f) = q/n_f$. Observe that

$$\rho(f^{dt}) = (1 + un_f)\rho(f) = \rho(f) + uq = \rho(f) \pmod 1.$$

Recall that a map g in $\mathcal{H}^+$ with fixed points is conjugate in $\mathcal{H}^+$ to all its powers. Let $g = f^{n_f}$. Then g is conjugate to g^{dt}. In other words, f^{n_f} is conjugate to $(f^{dt})^{n_f}$. Apply Theorem 2.3 to the maps f and f^{dt} to see that these two maps are conjugate. The second map f^{dt} is strongly reversible, because f^d is strongly reversible. Conjugacy preserves strong reversibility, therefore f is also strongly reversible.   □

We first prove a special case of Theorem 1.3 (ii).

LEMMA 5.3.   *Let f be an orientation preserving homeomorphism of $\mathbb{S}$ with rotation number $\frac{1}{n}$, for a positive integer n. Suppose that there is an open interval J such that the intervals $J, f(J), \ldots, f^{n-1}(J)$ are pairwise disjoint, and such that $\mathrm{fix}(f^n)$ is the complement of $J \cup f(J) \cup \cdots \cup f^{n-1}(J)$. Then f is strongly reversible by an orientation reversing involution that maps $f^k(J)$ to $f^{-k+1}(J)$ for each integer k.*

Proof.   Let R denote an anticlockwise rotation by $2\pi/n$. After conjugating f suitably, the function f and interval J may be adjusted so that $f^k(J) = R^k(J)$ for all integers k. Note that f^n has the same signature on each of the intervals $R^k(J)$ because, by Lemma 2.1 (i),

$$\Delta_{f^n}(f(x)) = \Delta_{f^{-1}f^nf}(x) = \Delta_{f^n}(x).$$

Let τ denote reflection in a line ℓ through the origin that bisects J. Thus τ fixes J as a set. Let σ denote reflection in a line through the origin that is π/n anticlockwise from ℓ. Then $R = \sigma\tau$. Orient J in an anticlockwise sense, and choose an increasing homeomorphism ϕ of J, without fixed points, such that $\tau\phi\tau = \phi^{-1}$ on J. This is possible because we can, by conjugation, consider J to be the real line and consider τ still to be a reflection, in which case ϕ can be chosen to be a translation. Now define an orientation preserving circle homeomorphism g to satisfy $g(x) = f(x)$, for x in $\mathrm{fix}(f^n)$, and $g(x) = R^{k+1}\phi R^{-k}(x)$, for x in $R^k(J)$. This means that g^n has the same signature on all of the intervals $R^k(J)$ (our choice of ϕ determines whether this signature is -1 or 1). By Theorem 2.3, g is conjugate to f. Also, one can check that $\sigma g \sigma = g^{-1}$ and $\sigma(g^k(J)) = g^{-k+1}(J)$ for each integer k. Since strong reversibility is preserved under conjugation, the result follows.   □

Proof of Theorem 1.3 (ii).   Suppose first that f is strongly reversible by an involution τ. Then f^{n_f} is also strongly reversible by τ. We must show that f^{n_f} is strongly reversible by an orientation reversing involution; thus we may assume that τ is orientation preserving. In this case, by Theorem 1.1, f is an involution. Therefore f^{n_f} is the identity, and as such it is reversible by any orientation reversing involution.

Conversely, suppose that there is an orientation reversing involution τ such that $\tau f^{n_f}\tau = f^{-n_f}$. Let p be a fixed point of f^{n_f}. There is an integer d that is coprime to n_f such that the distinct points $p, f^d(p), \ldots, f^{(n_f-1)d}(p)$ occur in that order anticlockwise around $\mathbb{S}$. The function $(f^d)^{n_f}$ is strongly reversible because it equals $(f^{n_f})^d$. If we can deduce that f^d is strongly reversible then it follows from Lemma 5.2 that f is strongly reversible. In other words, it is sufficient to prove the theorem when the points $p, f(p), \ldots, f^{n_f-1}(p)$ occur in that order around $\mathbb{S}$.

The map τf has a fixed point q which, by replacing τ with $f^k\tau f^{-k}$ for an appropriate integer k, we can assume lies in the interval $(f^{-1}(p), p]$. Either q is a fixed point of f^{n_f} (that is, a periodic point of f) or it is not. In the former case let $I = [q, f(q)]$ and define, for integers