A Fixed–point Theorem for Holomorphic Maps

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Abstract.

We consider the action on the maximal ideal space M of the algebra H of bounded analytic functions, induced by an analytic self–map of a complex manifold, X. After some general preliminaries, we focus on the question of the existence of fixed points for this action, in the case when X is the open unit disk, $\mathbf{D}$. We classify the fixed–point–free Möbius transformations, and we show that for an arbitrary analytic map from $\mathbf{D}$ into itself, the induced map has a fixed point, or it restricts to a fixed–point–free Möbius map on some analytic disk contained in M.

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The purpose of this paper is to present a new kind of fixed–point theorem. Let H^∞ denote the uniform algebra of all bounded analytic functions in the open unit disc, $\mathbf{D}$, and let M denote its maximal ideal space, or character space [B,G]. If $f : \mathbf{D} \to \mathbf{D}$ is holomorphic, then (as will be explained below) it induces a map $\check{f} : M \to M$ which "extends" f in a natural way. This induced map may have no fixed points in M. For instance, there are Möbius transformations f such that $\check{f}$ that has no fixed point. The main observation of this paper is that, in a sense, such Möbius transformations are the canonical fixed–point–free $\check{f}$'s. What happens is that for arbitrary f, there is a fixed point for $\check{f}$, or else there is an analytic disk $P \subset M$ that is mapped into itself by $\check{f}$, and on which $\check{f}$ is such a Möbius transformation.

In section 1 we will describe the map $\check{f}$ and its basic properties. Most of these are very well–known. In section 2 we present the main results.

1. The map $\check{f}$ and its basic properties.

The map $\check{f}$ may in fact be constructed in a rather more general setting, as follows.

Let X be any (connected) complex manifold, and let $H = H^\infty(X)$ denote the space of all bounded analytic functions on X. H may contain only the constant functions, depending on the nature of X. With the sup norm on X and pointwise operations, H becomes a Banach algebra. Since $\|g^2\| = \|g\|^2$ whenever $g \in H$, H is a uniform algebra on its maximal ideal space, M. As usual, we identify M with the space of characters, or nonzero algebra homomorphisms $\phi : H \to \mathbf{C}$. This space M may be regarded as a subset of the dual space H^* of H, and so inherits the metric of H^* (which is called the Gleason metric in this context), and the weak–star topology of H^*. We shall denote the Gleason distance between two homomorphisms ϕ and ψ by $\|\phi - \psi\|$. We shall have occasion to use the following fact:

Lemma 1. *The Gleason metric is weak–star lower semicontinuous on M, i.e.*

$$\lim_\alpha \|\phi_\alpha - \psi_\alpha\| \le \|\phi - \psi\|$$

whenever $\{\phi_\alpha\}$ and $\{\psi_\alpha\}$ are nets and $\phi_\alpha \to \phi$ and $\psi_\alpha \to \psi$.

Proof. This fact follows from the corresponding fact in dual Banach spaces. ■

Now let $f : X \to X$ be a holomorphic map. Then the map

$$\circ f : \begin{cases} H \to H \\ g \mapsto g \circ f \end{cases}$$

is an algebra homomorphism, hence we have a map

$$\check{f} : \begin{cases} M \to M \\ \phi \mapsto (g \mapsto \phi(g \circ f)) \end{cases}$$

The map $\check{f}$ is sometimes called the hull of f. This map is in fact just the restriction to M of the adjoint of the map $\circ f$. As a consequence, we obtain:

Lemma 2. *The induced map $\check{f}$ is a continuous both as a self–map of M with its Gleason metric topology, and as a self–map of M with its weak–star topology.*

Proof. Indeed, $\check{f}$ is a contraction with respect to the Gleason distance, and hence metric–continuous, and if $\phi_\alpha \to \phi$ weak–star, then

$$\check{f}(\phi_\alpha)(g) = \phi_\alpha(g \circ f) \to \phi(g \circ f) = \check{f}(\phi)(g)$$

whenever $g \in H$. ∎

When H separates points on X, we may regard X as a subset of M, and the map $\hat{f}$ as an extension of f.

It is an interesting question to ask for which X the map $\check{f}$ necessarily has a fixed point. There are obstructions in general, as is obvious from the example of rotation on an annulus. One general observation is this:

Lemma 3. *Let X, M, f be as above. There necessarily exists a point $\phi_0 \in M$ such that*

$$\|\check{f}(\phi_0) - \phi_0\| = \inf\left\{\|\check{f}(\phi) - \phi\| : \phi \in M\right\}.$$

Proof. Since $\check{f}$ is weak–star to weak–star continuous, the function

$$\phi \mapsto \|\check{f}(\phi) - \phi\|$$

is weak–star lower semicontinuous. Since M is weak–star compact, this function must attain its minimum. ∎

The infimum in Lemma 3 is necessarily less than 2. This follows from the fact that the Gleason distance between any two points of a complex manifold is less than 2 (— If H fails to separate points, then M has just one point and there is nothing to prove. In any case, as Gleason first observed, the relation $\phi \sim \psi$ if and only if $\|\phi - \psi\| < 2$ is an equivalence relation on M [G]; the Cauchy integral formula establishes the continuity of the Gleason distance near the diagonal of $X \times X$, and the transitivity of $\sim$ then shows that the distance cannot exceed 2 on $X \times X$).

The equivalence classes under the above relation $\sim$ on M are called the Gleason parts of H. Thus $\check{f}(\phi_0)$ lies in the same Gleason part as ϕ_0.

Corollary 4. *$\check{f}$ maps the Gleason part P of ϕ_0 into itself.*

Proof. This follows from the facts that $\check{f}$ contracts the Gleason distance, and the transitivity of $\sim$. ∎

Further, we note that by the minimality property of ϕ_0, we have

Corollary 5.

$$\|\check{f}(\check{f}(\phi_0)) - \check{f}(\phi_0)\| = \|\check{f}(\phi_0) - \phi_0\|.$$

∎

If ϕ_0 is not a fixed point of $\check{f}$, then this rigidity property is liable to impose strong restrictions on $\check{f}$; in particular, if there is analytic structure on the non–one–point parts of H, then it amounts to equality in the Schwarz lemma.

2. The unit disk.

Now we specialise to the case when $\dim X = \mathbf{D}$.

In this case, it is important not to confuse $\check{f}$ with the Gelfand map $\hat{f} : M \to \mathbf{C}$ defined by

$$\hat{f}(\phi) = \phi(f), \ \forall \phi \in M.$$

Let us denote the projection of M clos$\mathbf{D}$,

$$\phi \mapsto \phi(z \mapsto z)$$

by π. Then by applying Brouwer's fixed point theorem to dilations $f(rz)$ $(r < 1)$, it is easy to see that the function $\pi - \hat{f} : M \to \mathbf{C}$ has a zero, but this merely says that some point in some fibre of π is mapped into that fibre. It does not guarantee the existence of a fixed point.

We recall some facts about the structure of M. The principal reference for these is the celebrated paper of Hoffman [Annals].

The Gleason parts of H are of three main kinds. Those with more than one point have the structure of analytic disks. For such a part P there exists a bijection $h : P \to \mathbf{D}$ such that $\hat{f} \circ h^{-1} : \mathbf{D} \to \mathbf{C}$ is analytic whenever $f \in H$. We denote the union of all these disk parts by G (for good). The points on the Shilov boundary Sh(H) give one–point parts, and there are also other one–point parts. For instance, the zero set of the Gelfand transform of the singular inner function

$$z \mapsto \exp\left(\frac{z+1}{z-1}\right)$$

contains one–point parts and is disjoint from Sh(H) [Gam, p.162, ex.3; Garnett]. We denote the set of one–point parts off Sh(H) by B. The family of all hulls $\check{f}$ may be describes as the family of all weak–star continuous maps from M to M that are holomorphic on G. This statement is true because of the Corona Theorem of Carleson [Garnett] , which states that $\mathbf{D}$ is weak–star dense in M.

There is another way to classify the points of M, in terms of the way in which they may be approximated by points of $\mathbf{D}$. The points of $M \sim \mathbf{D}$ lie in the various fibres $M_\lambda = \pi^{-1}(\lambda)$ for $\lambda \in \mathbf{S}$. A point of $M\lambda$ is called nontangential if it is in the closure of a nontangential sector at λ, and horocyclic if it is in the closure of a horocyclic disk at λ. All such points lie in G. The points of G may be characterised as those which lie in the weak–star closures of interpolating sequences (— a sequence $\{x_n\} \subset \mathbf{D}$ such that $H|\{x_n\}$ is isomorphic to $l\infty$). At the other extreme, if $\{x_n\} \subset \mathbf{D}$ is a sequence that is an ϵ–net for the Gleason distance for some $\epsilon < 2$, then all non–disk points of M lie in the weak–star closure of $\{x_n\}$.

Theorem 1. *Let $f : \mathbf{D} \to \mathbf{D}$ be holomorphic and let $\check{f}$ be the induced self–map of the maximal ideal space M of $H = H^\infty(\mathbf{D})$. Then $\check{f}$ has a fixed point in M, or there is an analytic disk $P \subset M$ on which f acts as a Möbius map.*

In the sequel, we shall be more precise about the nature of the Möbius map.

Remarks. Some classical results are related to this theorem. First, if f is actually continuous up to the boundary, then by Brouwer's fixed–point theorem there is a fibre of π which is mapped into itself by $\check{f}$. For general f, the application of Brouwer's theorem to dilations of f shows that there exista a point which is mapped into its own fibre by $\check{f}$. This appears to be as far as Brouwer's theorem will take you. In 1926, Wolff that either f has a fixed point inside $\mathbf{D}$, or else there is a boundary point $\zeta \in \mathbf{S}$ such that each *horocyclic* disk at ζ is f–invariant, i.e. all disks internally tangent to $\mathbf{S}$ at ζ are mapped into themselves by f. See [Dineen, p.194] for this and generalisations to higher dimensions.

The induced map $\check{f}$ on M was defined and studied by Behrens in a series of papers from 1969 to 1972. (cf. [B in Vict, Stroyan + L, pp. 244-285]. He used methods of non–standard analysis, and he proved a number of results about fixed points for $\check{f}$. The nonstandard point of view is quite illuminating. If $z \in D^*$ is a point of the nonstandard open unit disk, and $f : D^* \to \mathbf{C}^*$ is an analytic function with $|f| < 1$, then we may define

$$(T(z))f = {}^circf(z),$$

the standard part of $f(z)$. $T(z)$ is then a complex homomorphism on H. The map T is a surjection from D^* onto M, and the points of $M \sim D$ correspond to points of D^* that are infinitesimally close to the unit circle. The hyperbolic metric extends to D^*, and the set of Gleason parts of H is in one–to–one correspondence with the set of hyperbolic galaxies of D^*.

Behrens main result on fixed points is this:

Behren's Theorem. *If $\check{f}$ (1) fixes 2 disk points (points of G) in distinct fibres,* **or** *(2) is inner and fixes a point of* $\mathbf{D}$ *and a point of Sh(H), then $f(z) \equiv z$.*

As regards the existence of fixed points, he observed the following:

(1) $\check{f}$ fixes a point of G if and only if

$$\inf_{\mathbf{D}} \|z - f(z)\|_{H^*} = 0.$$

(2) $\check{f}$ fixes a point of $G \cap M_\lambda$ if and only if f has angular derivative equal to 1 at λ, and if this happens then f fixes each nontangential point of M_λ and maps the weak–star closure of each tangent horodisk into itself.

(3) Each one–point part in the weak–star closure of a sequence of iterates $\{f^n(z)\}$ is a fixed point for $\check{f}$.

(4) However, if the sequence $\{f^n(z)\}$ is interpolating, then no point of its weak–star closure is a a fixed point for $\check{f}$.

He also showed that the hull of $z \mapsto z^n$ fixes only 0, and appears to assert that the hull of $\frac{2-z}{1-2z}$ does have fixed points. This latter assertion [Vic, p.] is probably a misprint, as will appear.

Observations (3) and (4) are also easily seen by standard arguments.

The proof we give of Theorem 1 does not require any of Behren's results. It uses only the results quoted above in section 1, and the part structure of H. However, we shall make use of Behren's results and the nonstandard approach to prove another result which allows us to sharpen the conclusion of Theorem 1.

Proof of Theorem 1.

Let ϕ_0 be a point, as in Lemma (1.3), at which $\|\check{f}(\phi) - \phi\|$ attains its minimum on M, and suppose it could happen that $\check{f}(\phi_0) \neq \phi_0$. Let P be the Gleason part of ϕ_0, which is mapped into itself by $\check{f}$ (Cor.(1.4)). Then P is an analytic disc, so there is a bijection $h : \mathbf{D} \to P$ such that

$$\hat{g} \circ h : \mathbf{D} \to \mathbf{C}$$

is analytic whenever $g \in H$, and $\check{f}$ is an analytic map of P into P, in the sense that the map

$$k = h^{-1} \circ f \circ h : \mathbf{D} \to \mathbf{D}$$

is analytic. But this means that Cor. (1.5) gives equality in the Schwarz Lemma for k, so that k is a Möbius transformation of $\mathbf{D}$. ∎

Now, consider the case when f is a Möbius transformation. In analyzing this, it will sometimes be convenient to switch from the disk to the (conformally–equivalent) upper half– plane, $\mathbf{H}$.

For the present purpose, the Möbius self–maps of the disk may be divided into four classes:

I: the identity map, z.

II: those having an internal fixed point (and the other off the closed disk). The internal fixed point is attracting.

III: those having two fixed points on the circle (in the ordinary sense) One fixed point attracts, the other repels. This type is typefied by

$$z \mapsto \frac{2 - z}{1 - 2z}$$

on $\mathbf{D}$, or $z \mapsto z/2$ on $\mathbf{H}$.

IV: those having a single degenerate fixed point on the circle (and no other fixed point on the sphere). This is typefied by

$$\frac{z + i(z - 1)}{1 + i(z - 1)}$$

on $\mathbf{D}$, or $z \mapsto z + 1$ on $\mathbf{H}$.

The type of a Möbius map is evidently a conjugacy invariant.

The hull of a Möbius map is a bijection of M onto itself, and it is an isometry with respect to the Gleason distance. Types I and II fix points in $\mathbf{D}$. The hull $\check{f}$ of an f type III or type IV permutes the fibre of each fixed point of f on $\mathbf{S}$, but does not fix all points in such fibres. In fact, each sequence of iterates for either type is an interpolating sequence, and tends to a fixed point of f on the circle, and we know that no weak–star limit of an interpolating sequence of iterates is fixed by $\check{f}$.

Theorem 2. *If f is a Möbius map, then f is of Type III if and only if $\check{f}$ has no fixed point.*

Proof. Type I or II have fixed points in $\mathbf{D}$.

The existence of fixed points for type IV is most readily seen by transferring to the upper half–plane and noting that when the points $x_n = ni$ in the upper half–plane are mapped by $f(z) = z + 1$, we get by a short calculation that

$$\|f(x_n) - x_n\| \leq \frac{1}{n} \to 0.$$

Thus each weak–star accumulation point of $\{x_n\}$ is a fixed point for $\check{f}$, by Lemmas 1 and 2.

It remains to see that type III maps have no fixed points.

It is sufficient to deal with the maps on $\mathbf{H}$ given by $f_a(z) = az$ for $a > 0$, $a \neq 1$.

The only fibres which intersect their images under $\check{f}_a$ are M_0 and M_∞. The case of M_∞ is equivalent to the case of M_0 for the map $z \mapsto a^{-1}z$, so it is sufficient to show that $\check{f}_a$ has no fixed point in M_0.

Now f_a has angular derivative equal to a at 0, so by Behren's observation (2), $\check{f}_a$ fixes no point of G.

Let $\phi \in M_0 \sim G$. Then there is a point $\zeta \in \mathbf{H}^*$, the nonstandard upper half–plane which is mapped to ϕ by the map T. Since ϕ is not a nontangential point, we have $\zeta = \xi + i\eta$, with η/ξ infinitesimal, i.e. the argument of ζ is infinitesimally close to 0 or to π. Now [S+L] the nonstandard version of $\check{f}$ generates $\check{f}_a$, in the sense that

$$\check{f}(T(z)) = T(f(z)), \qquad \forall z \in \mathbf{H}^*.$$

Let d denote the hyperbolic distance on $\mathbf{H}$. Then

$$d(\zeta, f_a(\zeta)) \geq \frac{(1-a)|\xi + i\eta|}{\max\{\eta, a\eta\}}$$

and this is infinite. Thus $f_a(\zeta)$ lies outside the hyperbolic galaxy of ζ, hence $\check{f}_a(\phi)$ lies outside the part of ϕ. In particular, $\check{f}_a(\phi) \neq \phi$.

Thus $\check{f}_a$ has no fixed point in M. ∎

Corollary 3. *Under the hypotheses of Theorem 1, $\check{f}$ has a fixed point, or it restricts to a Type III Möbius transformation on some analytic disk in M.*

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