Approximation and extension in normed spaces of infinitely differentiable functions

J.F. Feinstein, H. Lande and A.G. O'Farrell*

Abstract We consider questions of rational and polynomial approximation, and related extension questions, for various normed spaces of infinitely differentiable functions on perfect compact subsets of the complex plane $\mathbb{C}$ and the real line $\mathbb{R}$. We obtain an approximation theorem for compact planar sets which are, in a precise sense, locally radially–self–absorbing. All smoothly–bounded compact sets are of this type. We give a variety of results and counterexamples on extension, mainly in the one-dimensional case. We also prove polynomial approximation theorems for totally–disconnected sets, linear sets, and some others.

1. Introduction.

Classes of infinitely differentiable functions have been extensively studied (see, for example, [BA], [BR], [CM], [K], [M], [W]), but these studies addressed questions concerning nuclear or Fréchet spaces. Less work has been done on the related normed spaces, which require more delicate analysis.

In this paper, we consider the normed spaces $D(K, M)$, where K is a perfect, compact plane set, and $M = \{M_n\}_{n\geq 0}$ is a sequence of positive real numbers. We shall give the definitions of these spaces of functions in Section 2. They were initially studied by Dales, Davie and McClure, with particular reference to the case where $D(K, M)$ is a complete normed algebra. In [DD] Dales and Davie gave some sufficient conditions for the space to be complete, examined its character space, and investigated peak points, and quasi-analyticity. They also gave a negative answer to a question of Curtis on the automatic continuity of functions which operate on a natural Banach function algebra. In [DM], Dales and McClure proved that, if K is the closed unit disk, then the polynomials are always dense in $D(K, M)$. A similar result for rational approximation on the annulus has been obtained by Honari [H].

In the one-dimensional case, some results on polynomial approximation were Obtained by one of us in [O1]. It was shown under very mild conditions on M that the polynomials are dense in $D(I, M)$, where I is a closed interval.

We say that a set $E \subset \mathbb{R}^d$ is *radially self-absorbing* if for each $r > 1$, we have

$$E \subset \text{int}(rE),$$

where rE is the dilation of E defined by

$$rE = \{rx : x \in E\}.$$

Such sets are star–shaped with respect to the origin. In Section 3 we shall see that, if K is a compact plane set which is radially self-absorbing, then the restrictions to K of functions

* This work was supported in part by EOLAS grant SC/90/070

holomorphic on a neighbourhood of K are dense in $D(K,M)$. In the Banach *algebra* case, and provided that all the rational functions with poles off K belong to $D(K,M)$, it will follow that the rational functions, and indeed the polynomials, are dense in $D(K,M)$.

By a localization argument, we obtain a similar result for compact sets K that are *locally radially self–absorbing* in the sense that for each point $a \in K$ there is a closed neighbourhood N of a and a point $b \in K \cap N$ such that the set

$$(K \cap N) - b = \{x - b : x \in K \cap N\}$$

is radially self–absorbing. The result we prove is that if K is locally radially self–absorbing, then those functions in $D(K,M)$ which have suitable global extensions can again be approximated by functions holomorphic in a neighbourhood of K.

It is readily seen that a compact $K \subset \mathbb{C}$ is locally radially self–absorbing whenever K has dense interior and the boundary of K consists of a finite union of pairwise disjoint, smooth Jordan curves.

Our method of attack on the approximation problems raises the question of the existence of suitable extensions. Little has been done in this area, for the normed spaces. In Section 4, we give a variety of results on extension questions in one dimension. We show that the answers obtained when $M_n = (n!)^\alpha$ for some α are different from those obtained when M_n is very rapidly increasing, e.g. $M_n = 2^{2^n}$. In Section 5, we prove some more approximation theorems. In particular, we show that the polynomials are dense in $D(K,M)$ for many M and each compact perfect set $K \subset \mathbb{R}$.

2. Preliminary definitions and results.

We denote the space of infinitely–differentiable complex-valued functions on $\mathbb{R}^d$ by $C^\infty(\mathbb{R}^d)$. We identify $\mathbb{C}$ with $\mathbb{R}^2$, so $C^\infty(\mathbb{C})$ consists of those functions from $\mathbb{C}$ to $\mathbb{C}$ having partial derivatives of all orders, whether or not they are analytic. We denote the n-th Fréchet derivative of a function $f : \mathbb{R}^d \to \mathbb{C}$ by $D^n f$. For each $a \in \mathbb{R}^d$, $D^k f(a)$ is a symmetric k–linear function on $(\mathbb{R}^d)^k$. We need to fix a convention about the norms of such objects, and record a few observations.

If $A : (\mathbb{R}^d)^k \to \mathbb{C}$ is symmetric and k–linear, then we let

$$|A| = \sup\{|\langle (x_1, \ldots, x_k), A\rangle| : |x_i| \le 1,\ \forall i\}.$$

The symmetric product of a symmetric i–linear function A and a symmetric j–linear function B is defined by

$$\langle (x_1, \ldots x_{i+j}), A \odot B\rangle = \frac{1}{(i+j)!} \sum_{\sigma \in S_{i+j}} \langle (x_{\sigma(1)}, \ldots, x_{\sigma(i)}), A\rangle \langle (x_{\sigma(i+1)}, \ldots, x_{\sigma(i+j)}), B\rangle.$$

Here S_{i+j} denotes the symmetric group on $i+j$ symbols.

We have

$$|A \odot B| \le |A| \cdot |B|.$$

Leibnitz' rule gives

$$D^k(f \cdot g)(a) = \sum_{j=0}^{k} \binom{k}{j} D^j f(a) \odot D^{k-j} g(a),$$

so

$$|D^k(f \cdot g)(a)| \le \sum_{j=0}^{k} \binom{k}{j} |D^j f(a)| \cdot |D^{k-j} g(a)|.$$

Also

$$|D^k(\partial^j \phi)(a)| \le |D^{k+|j|}\phi(a)|,$$

whenever j is a multi–index.

Let S be a set, let V be a normed vector space, and let $f : S \to V$ be a function. Then the *uniform norm of f on S*, $\|f\|_S$ is defined by

$$\|f\|_S = \sup\{|f(x)| : x \in S\}.$$

Let $\{M_n\}_{n=0}^{\infty}$ be a sequence of positive numbers. For $1 \le r < \infty$ we set

$$E(d, r, M) = \{f \in C^\infty(\mathbb{R}^d) : \sum_{n=0}^{\infty} (M_n^{-1} \|D^n f\|_{\mathbb{R}^d})^r < +\infty\}.$$

$E(d, r, M)$ is a Banach space with respect to the norm

$$\|f\| = \left(\sum_{n=0}^{\infty} (M_n^{-1} \|D^n f\|_{\mathbb{R}^d})^r \right)^{\frac{1}{r}}.$$

We define $E(d, \infty, M)$ to be the corresponding Banach space defined using suprema.

We abbreviate $E(d, M) = E(d, 1, M)$.

Now let K be a perfect, compact subset of $\mathbb{C}$. We say that a function $f : K \to \mathbb{C}$ is *complex-differentiable* at a point $a \in K$ if the limit

$$f'(a) = \lim_{z \to a,\ z \in K} \frac{f(z) - f(a)}{z - a}$$

exists. We call $f'(a)$ the *complex derivative* of f at a. Using this concept of derivative, we define the terms *complex–differentiable on K*, *continuously complex–differentiable on K*, and *infinitely complex–differentiable on K* in the obvious way. We denote the n-th complex derivative of f at a by $f^{(n)}(a)$. We then define normed spaces $D(K, r, M)$ of infinitely complex–differentiable functions on K with norms corresponding to those in the spaces $E(2, r, M)$: f belongs to $D(K, r, M)$ if it is infinitely complex–differentiable on K and the l_r norm of the sequence

$$\|f^{(n)}\|_K / M_n$$

is finite.

If a compact set K is a subset of $\mathbb{R}$, then we regard it as lying on the real axis of $\mathbb{C}$ and interpret $D(K,r,M)$ accordingly. For such sets, "complex differentiablity"on K just means differentiability is the usual (real–variable) sense.

All polynomials when restricted to K belong to each $D(K,r,M)$. When $1 \le r \le \infty$ the space $D(K,r,M)$ includes all the rational functions with poles off K if and only if

$$\lim_{n\to\infty} \left(\frac{n!}{M_n}\right)^{\frac{1}{n}} = 0. \tag{1}$$

We say that M is a *nonanalytic sequence* if (1) holds.

We abbreviate $I = [-1,1]$.

We abbreviate $D(K,M) = D(K,1,M)$.

If the sequence M satisfies $M_0 = M_1 = 1$ and there exists a constant $\kappa(M) > 0$ such that

$$\frac{M_n}{M_k M_{n-k}} \ge \kappa^{-1} \cdot \binom{n}{k} \tag{2}$$

for all non-negative integers k, n with $k \le n$ then $D(K,M)$ is a normed algebra. More precisely, the equivalent norm $f \mapsto \kappa\|f\|$ is an algebra norm on $D(K,M)$ (i.e. it is submultiplicative). In general $D(K,M)$ is incomplete.

A few words about the precise relation of Frechet derivatives and complex derivatives, and the spaces $D(K,r,M)$ and $E(2,r,M)$ are in order. If $f'(a)$ and $Df(a)$ both exist, then for $u \in \mathbb{C}(=\mathbb{R}^2)$,

$$\langle u, Df(a)\rangle = f'(a)u,$$

(where the lhs denotes the action of the linear function $Df(a)$ on the 2-dimensional vector u, and the rhs denotes the product of the complex numbers $f'(a)$ and u). Similarly, for higher derivatives, we have

$$\langle (u_1,\ldots,u_k), D^k f(a)\rangle = f^{(k)}(a)u_1\cdots u_k,$$

and

$$\|D^k f(a)\| = |f^{(k)}(a)|.$$

If f belongs to $E(2,r,M)$, then the restriction $f|K$ belongs to $D(K,r,M)$ provided f is complex–differentiable on K. To be complex–differentiable on K, f must be holomorphic on the interior of K, and may also have to satisfy other conditions. If the interior of K is dense in K, then it is sufficient that f be holomorphic on int$K$. At the other extreme, if K is a C^∞ nonsingular curve, then each element of $E(2,r,M)$ restricts to a member of $D(K,r,M)$. In general, a function $f \in E(2,r,M)$ will not restrict to an element of $D(K,r,M)$ unless it satisfies special conditions at many points of $K \sim \text{int}K$. These conditions include the point Cauchy–Riemann condition

$$D^k\left(\frac{\bar\partial f}{\partial \bar z}\right)(a) = 0 \qquad \forall k,$$

at each point a for which the germ of K at a does not fit into some C^∞ curve.

We say that M is *an algebra sequence* if (2) holds, and we call κ an algebra constant for M. For such a sequence, $E(2, M)$ becomes a Banach algebra (i.e. a complete normed algebra, with submultiplicative norm, cf. [R]) when endowed with the above equivalent norm.

Note that the ("Gevrey") sequence $M_n = (n!)^\alpha$ is an algebra sequence for $\alpha \geq 1$, and is nonanalytic for $\alpha > 1$.

If M is an algebra sequence, then we call it a *non–quasianalytic* sequence if

$$\sum_n \frac{M_n}{M_{n+1}} < +\infty.$$

One can show that each non–quasianalytic algebra sequence is non–analytic.

The classical Denjoy–Carleman theorem [R, Chapter 19] assures us that if M is a non–quasianalytic sequence, then there exist functions $f \in E(1, M)$ which have compact support but are not identically zero. The same is true of $E(d, m)$ for each d. In fact, if $f \in E(1, M)$ is not identically zero and has compact support, then each of the functions $(x_1, \ldots, x_d) \mapsto f(x_i)$ belongs to $E(d, M)$, and their product is not identically zero and has compact support. It follows easily that each open covering of $\mathbb{C}$ has a subordinate partition of unity consisting of elements of $E(2, M)$.

A compact plane set K is *uniformly regular* in the sense of Dales and Davie if, for all z, w in K, there is a rectifiable arc in K joining z to w, and the metric given by the geodesic distance between points of K is uniformly equivalent to the Euclidean metric on K.

If K is a compact plane set which is a finite union of uniformly regular sets, and M is a sequence of positive real numbers, then $D(K, M)$ is complete, and hence is a Banach space [cf. DD, Theorem 1.6]. If $K \subset \mathbb{R}$, is compact then $D(K, M)$ is complete if and only if K has only a finite number of connected components.

The following question appears to be open: Are the rational functions with poles off K dense in $D(K, M)$ whenever K is a perfect compact plane set and M is a nonanalytic sequence?

Given a perfect set $K \subseteq \mathbb{C}$, a number r, and a sequence M, we will also be interested in the subspace $D_1(K, r, M)$ which consists of those elements of $D(K, r, M)$ which have extensions in $C^\infty(\mathbb{C})$. Since K is perfect, these correspond precisely to the appropriate subset of the C^∞ Whitney jets of functions on K. We shall see that $D_1(K, r, M)$ may be a proper subset of $D(K, r, M)$.

Given r and M we have the sequence space $l_r(M_n^{-1})$ consisting of those sequences $(\alpha_n)_{n=0}^\infty$ such that $(\alpha_n / M_n) \in l_r$. We also have a norm decreasing linear map

$$\rho_r : E(2, r, M) \longrightarrow l_r(M_n^{-1})$$

defined by

$$\rho_r(f) = (f^{(n)}(0))_{n=0}^\infty.$$

We may now ask the question: for which sequences (M_n) and which r are the maps ρ_r onto? Similarly, for $K \subseteq \mathbb{C}$ we may ask which functions in $D(K,r,M)$ have extensions in the space $E(2,r,M)$?

In Section 4, we consider extension from compact subsets $K \subset \mathbb{R}$, and we show the following:

• (1) There are nonquasianalytic sequences M such that $\rho_1 : E(1,M) \to l_1(M_n^{-1})$ is not surjective. An example is $M_n = (n!)^\alpha$, $\alpha > 1$.

• (2) For all sequences M that grow sufficiently rapidly, the map ρ_1 *is* surjective (and there is a linear lifting). An example is $M_n = 2^{2^n}$.

• (3) For each sequence M and each $r \geq 1$ there exists a perfect set K such that not all functions $f \in D_1(K,r,M)$ have extensions in $E(1,r,M)$.

• (4) For each sequence M there is a perfect set K such that $0 \in K$ and each sequence $\{\alpha_n\} \in l_1(M_n^{-1})$ is obtained as $\{f^{(k)}(0)\}$ for some $f \in D(K,M)$.

In Section 5, we present a method which reduces rational approximation in D(K,M) spaces to the case of connected K, and we use it to obtain approximation theorems for arbitrary sets on the line and for totally-disconnected sets in the plane.

3. Approximation results for plane sets.

(3.1) We begin with a result on holomorphic approximation for sets which are radially self–absorbing.

Lemma 3.1. *Let K be a compact radially self–absorbing plane set. Then, for $1 \leq r < \infty$ and any sequence M, the set of those $f \in D(K,r,M)$ which are restrictions of functions holomorphic in a neighbourhood of K is dense in $D(K,r,M)$.*

Remarks. Every such set K is perfect, starshaped, and has $0 \in \operatorname{int}(K)$. In fact, such sets K have the form $\{re^{i\theta} : 0 \leq \theta \leq 2\pi, 0 \leq r \leq R(\theta)\}$ for some continuous, positive, real-valued function R on $[0,2\pi]$ with $R(0) = R(2\pi)$. Such sets K need not be uniformly–regular, or even finite unions of uniformly–regular sets. If the function R is piecewise smooth (or equivalently, if bdy$K$ is piecewise smooth), then K is a finite union of uniformly–regular sets, so $D(K,r,M)$ is complete.

Proof. Let $f \in D(K,r,M)$. For $z \in K$ and $n \in \mathbb{N}$, set $f_n(z) = f(\frac{n}{n+1}z)$. It is enough to show that $f_n \to f$ in $D(K,r,M)$. First note that, for all n, k,

$$\|f_n^{(k)}\|_K \leq \|f^{(k)}\|_K,$$

so that f_n is in $D(K,r,M)$. Clearly $\|f_n^{(k)} - f^{(k)}\|_K \to 0$ as $n \to \infty$, and so the result follows by dominated convergence. ∎

Corollary 3.2. *Let K be a compact radially self–absorbing plane set for which $D(K,M)$ is complete. Suppose that M is a nonanalytic algebra sequence. Then the polynomials are dense in $D(K,M)$.*

Proof. Let Z denote the coordinate functional on K, set $A = D(K,M)$ and let B denote the closure in A of the polynomials. Then the spectrum of Z in A is just K, as is seen by using the rational functions $1/(Z-a)$. Since $\mathbb{C} \sim K$ is connected, it follows that the spectrum of Z in B is also K. Applying the holomorphic functional calculus to $Z \in B$, we see that all restrictions to K of functions holomorphic on a neighbourhood of K lie in B. The rest follows from the case $r = 1$ of the lemma. ∎

Remark. It is possible to drop the hypothesis that $D(K, M)$ be complete. See section 5 below.

Next we give a version of the lemma for $E(2, M)$ spaces. This result is only interesting in the non-quasianalytic case.

Corollary 3.3. *Let K be a compact radially self–absorbing plane set. let $1 \le r < \infty$ and let M be any sequence. Suppose that $f \in E(2, r, M)$ has compact support, and is holomorphic on int(K). Then f may be approximated in $E(2, r, M)$ by functions that have compact support and are holomorphic on a neighbourhood of K.*

Proof. As in the proof of Lemma 3.1, we define $f_n(z) = f(\frac{n}{n+1}z)$, for $z \in \mathbb{C}$ and $n \in \mathbb{N}$. Again, for all k, $\|{f_n}^{(k)}\|_K \le \|f^{(k)}\|_K$, and it is still true that $f_n^{(k)} \to f^{(k)}$ on K as $n \to \infty$, because the support of f is bounded. The result follows, as before. ∎

(3.2) We move now to more general compact sets, and our main approximation theorem.

Theorem 3.4. *Let K be a compact subset of $\mathbb{C}$ that is locally radially self–absorbing. Let M be a non–quasianalytic algebra sequence. Let $f \in E(2, M)$ be analytic on int$K$. Then there exists a sequence $F_n \in E(2, M)$ of functions that are holomorphic on a neighbourhood of K, converging to f in $E(2, M)$ norm.*

The proof of this theorem is based on the use of the Vitushkin localisation operator T_ϕ. This familiar tool of approximation theory [O2] is defined by the formula

$$T_\phi f = C(\phi \cdot \bar{\partial} f),$$

where ϕ ranges over test functions (C^∞ functions having compact support) and f ranges over complex–valued distributions on $\mathbb{C}$. Here C denotes the Cauchy transform:

$$Cg = \left(\frac{1}{\pi z}\right) * g,$$

and

$$\bar{\partial} f = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right).$$

This operator has the property that

$$\bar{\partial} T_\phi f = \phi \cdot \bar{\partial} f,$$

so that $T_\phi f$ is analytic wherever f is analytic and also off the support of ϕ.

We can use the T_ϕ operator with $\phi \in E(2,M)$. The fact that M is non–quasianalytic entails that we can find a partition of unity subordinate to any given covering of the plane, and consisting of elements of $E(2,M)$.

Let us denote by M^- the sequence M shifted one place right:

$$M_n^- = \begin{cases} M_{n-1}, n > 0, \\ 1, n = 0. \end{cases}$$

M^- shares the property of non–quasianalyticity with M. Observe that

$$\|\bar\partial f\|_{E(2,M)} \leq \|f\|_{E(2,M^-)},$$

since $|D^k \bar\partial f| \leq |D^{k+1} f|$.

Lemma 3.5. *Let M be a non–quasianalytic algebra sequence. Let $f \in E(2,M)$ and $\phi \in E(2,M^-)$. Suppose that ϕ has compact support. Then $T_\phi f$ belongs to $E(2,M)$.*

Proof. Since C inverts $\bar\partial$ on compactly–supported distributions, we get

$$\begin{aligned} C(\phi \cdot \bar\partial f) &= C\left(\bar\partial(\phi f) - (\bar\partial\phi) \cdot f\right) \\ &= \phi \cdot f - C\left((\bar\partial\phi) \cdot f)\right). \end{aligned}$$

Since M is an algebra sequence,

$$\|\phi \cdot f\|_{E(2,M)} \leq \kappa \cdot \|\phi\|_{E(2,M)} \cdot \|f\|_{E(2,M)}.$$

where κ is an algebra constant for M. Thus

$$\begin{aligned} \left\|(\bar\partial\phi) \cdot f\right\|_{E(M)} &\leq \kappa \cdot \left\|\bar\partial\phi\right\|_{E(M)} \cdot \|f\|_{E(M)} \\ &\leq \kappa \cdot \|\phi\|_{E(2,M^-)} \cdot \|f\|_{E(M)}. \end{aligned}$$

Now let $\Delta(\phi)$ be a minimal closed disk containing the support of ϕ and let $d = d(\phi)$ be the diameter of $\Delta(\phi)$.

Let $g = (\bar\partial\phi) \cdot f$, and observe that g has support in $\Delta(\phi)$. We wish to estimate the sup norm of $D^k(Cg)$.

For $w \in \mathbb{C}$ we have

$$|Cg(w)| \leq \|g\|_{\mathbb{C}} \cdot \int_{\Delta(\phi)} \frac{dxdy}{|x+iy-w|}.$$

It is not hard to see that the rhs is maximized when w is the centre of $\Delta(\phi)$, so we obtain

$$\|Cg\|_{\mathbb{C}} \leq 2d \cdot \|g\|_{\mathbb{C}},$$

and since

$$D^k\left(Cg\right) = C(D^k g),$$

we obtain

$$\left\|D^k(Cg)\right\|_{\mathbb{C}} \leq 2d \cdot \|D^k g\|_{\mathbb{C}},$$

so

$$\begin{aligned}\left\|C((\bar{\partial}\phi)\cdot f)\right\|_{E(M)} &\leq 2d \cdot \left\|(\bar{\partial}\phi)\cdot f\right\|_{E(M)} \\ &\leq 2d \cdot \kappa \cdot \|\phi\|_{E(2,M^-)} \cdot \|f\|_{E(M)}.\end{aligned}$$

Putting it all together, we obtain

$$\begin{aligned}\|T_\phi f\|_{E(M)} &\leq \left\{\|\phi\|_{E(M)} + 2d\cdot\kappa\cdot\|\phi\|_{E(2,M^-)}\right\}\|f\|_{E(M)} \\ &\leq (1+2\kappa d)\cdot\|\phi\|_{E(2,M^-)}\cdot\|f\|_{E(M)}.\end{aligned}$$

This proves the lemma. ∎

Proof of Theorem 3.4. Suppose that M, K and f are as in the statement. For each $a \in K$ there exists a compact neighbourhood N of a and a point $b \in N$ such that $(N \cap K) - b$ is radially self–absorbing. We can cover K by a finite number of such neighbourhoods, $N_1, \ldots, N_p$, and select functions $\phi_1, \ldots, \phi_p$ such that each ϕ_j belongs to $E(2, M^-)$, has compact support contained in $\text{int}N_j$, and $\sum \phi_j = 1$ on a neighbourhood of K. Setting $f_j = T_{\phi_j} f$ and $g = f - \sum f_j$, we then find that f_j is analytic on $\text{int}K$ and off $\text{spt}\phi_j$, and g is analytic on a neighbourhood of K. By the Lemma, each f_j belongs to $E(2, M)$, and hence so does g.

Thus it remains to see that each f_j may be approximated in $E(2, M)$ norm by functions that are holomorphic on a neighbourhood of K. This is readily done by using the dilation method of the proof of Lemma 3.1. The only point to check is that the dilates by factors slightly greater than 1 remain holomorphic on a neighbourhood of K, and not just $K \cap N_j$. The f_j are holomorphic off $\text{spt}\phi_j \sim \text{int}K$, and this is a compact set contained in $\text{int}N_j$. Its intersection with K is contained in $K \cap \text{int}N_j$. Thus slight dilation from a suitable point in $N_j \cap \text{int}K$ will force the singular support of f_j to miss K. ∎

Remarks.

1. We note that the restriction to K of the sequence F_n of the statement will converge in $D(K, M)$ norm to f, and that it follows that f lies in the closure of the rational functions in $D(K, M)$. If X is polynomially–convex, f then lies in the closure of the polynomials in $D(K, M)$.

2. It is possible that, for every perfect compact set K and every sequence M that a dense subset of elements of $D(K, M)$ extend to the appropriate $E(2, M)$ space. In this case, the above will give us rational approximation results for all K and appropriate M. Little was known about extensions of this type. We shall address the problem of extensions from $D(K, M)$ in the next section, and consider extensions from a dense subset in section 5.

3. The result applies to all compact sets with dense interior and merely *piecewise–smooth* boundary, provided the boundary is free of cusps. Certain kinds of cusps can be accommodated, but there are examples both of inward and outward cusps which cause the hypotheses to fail.

4. Extensions in one dimension.

(4.1) In this section we suppose that K is a perfect compact subset of $\mathbb{R}$ and that $0 \in K$.

Let $T_y^j g$ denote the j–th order Taylor polynomial of g about y. This has nothing to do with the T_ϕ operator.

The space $W(K,r,M)$ is defined to be the subset of $D(K,r,M)$ consisting of those functions f such that the seminorm

$$\|f\|'_{K,M} = \sup_{n\geq 0}\left\{\frac{1}{M_{n+1}}\cdot\sup_{j\geq 0}(n-j+1)!\left\|\frac{|D^j f(x)-((T_y^{n-j})D^j f)(y)|}{|x-y|^{n-j+1}}\right\|_{K\times K\sim\text{diagonal}}\right\} < +\infty.$$

We give this space the norm obtained by adding the above seminorm to the $D(K,r,M)$ norm. It is clear from Taylor's theorem that restrictions to K of elements of $E(1,r,M)$ always belong to $W(K,r,M)$. Unlike $D(K,r,M)$, the space $W(K,r,M)$ is complete for all compact perfect K.

In considering extensions from $D(K,r,M)$, it is natural to look at the following chain:

$$E(1,r,M)\to W(K,r,M)\to D(K,r,M)\to l_r(M^{-1}).$$

Here the first map is restriction to K, the second is an inclusion map, and the third is the map of a function to the sequence of its derivatives at 0. The composition of all the maps is ρ_r. We are focussing on the composition of the first two maps, and the relation between its image, $D_2(K,r,M)=E(1,r,M)\cap D(K,r,M)$ and $D(K,r,M)$. (Recall that $D_1(K,r,M)=C^\infty(\mathbb{R})\cap D(K,r,M)$, so that

$$D_2(K,r,M)\subset D_1(K,r,M)\subset D(K,r,M).$$

We begin by showing that ρ_r need not be surjective.

4.1 Theorem. *Suppose that M satisfies the following condition:*

$$\sup_k \inf_n \frac{M_{n-1}}{M_n}\cdot\left(\frac{M_{n+k}}{M_n}\right)^{\frac{1}{k}} < +\infty.$$

Let $1\leq r\leq\infty$. Then the restriction map $\rho_r : E(1,r,M)\to l_r(M^{-1})$ is not onto.

Remark. The conditions of the theorem are satisfied by the Gevrey sequences $(n!)^\alpha$ for all α. To see this, we observe that

$$\begin{aligned}&\inf_n \frac{1}{n^\alpha}\left((n+1)^\alpha\cdots(n+k)^\alpha\right)^{\frac{1}{k}}\\ &\quad=\inf_n\left(\left(1+\frac{1}{n}\right)\cdots\left(1+\frac{k}{n}\right)\right)^{\frac{\alpha}{k}}\\ &\quad\leq\inf_n\exp\left(\frac{\frac{1}{2}(k+1)\alpha}{n}\right)=1.\end{aligned}$$

Proof of Theorem. Choose $B > 1$ such that

$$\sup_k \inf_n \frac{M_{n-1}}{M_n} \cdot \left(\frac{M_{n+k}}{M_n} \right)^{\frac{1}{k}} < B.$$

Suppose, for a contradiction, that ρ_r is onto. By the open mapping theorem, there exists $C > 1$ such that, for each $\alpha = (\alpha_n) \in l_r(M_n^{-1})$ there exists $f \in E(1, r, M)$ with $\|f\| \leq C\|\alpha\|$ and such that $f^{(n)}(0) = \alpha_n$ for $n \geq 0$.

Now set $A = 4BC$. Choose $k \in \mathbb{N}$ such that

$$\frac{(AB)^{k-1}}{(k-1)!} < \frac{1}{A^2}.$$

Choose $n \in \mathbb{N}$ such that

$$\frac{M_{n-1}}{M_n} \left(\frac{M_{n+k}}{M_n} \right)^{\frac{1}{k}} < B,$$

and so

$$M_{n+k} < B^k \left(\frac{M_n}{M_{n-1}} \right)^k \cdot M_n.$$

By our assumption, there exists $f \in D(r, M)$ with $\|f\| \leq C$ and such that $f^{(t)}(0) = M_n \delta_{t,n}$ for all t. For this f, we have the trivial estimate $\|f^{(n+k)}\|_{\mathbb{R}} \leq CM_{n+k}$. Applying Taylor's theorem to the function $f^{(n+1)}$, and using the above, we find that, for $x \in [0, AM_{n-1}/M_n]$, we have

$$\begin{aligned}
|f^{(n+1)}(x)| &\leq \frac{1}{(k-1)!} \left(\frac{AM_{n-1}}{M_n} \right)^{k-1} \|f^{(n+k)}\|_{\mathbb{R}^d} \\
&\leq \frac{1}{(k-1)!} \left(\frac{AM_{n-1}}{M_n} \right)^{k-1} CM_{n+k} \\
&\leq \frac{C}{(k-1)!} \left(\frac{AM_{n-1}}{M_n} \right)^{k-1} B^k \left(\frac{M_n}{M_{n-1}} \right)^k M_n \\
&\leq \frac{BC}{A^2} \frac{M_n^2}{M_{n-1}} = \frac{M_n^2}{4AM_{n-1}}.
\end{aligned}$$

Since $f^{(n)}(0) = M_n$ it follows that, for all $x \in [0, AM_{n-1}/M_n]$,

$$f^{(n)}(x) \geq M_n - \frac{M_n^2}{4AM_{n-1}} \cdot \frac{AM_{n-1}}{M_n} = \frac{3}{4} M_n$$

and hence

$$\begin{aligned}
f^{(n-1)} \left(\frac{AM_{n-1}}{M_n} \right) &\geq \frac{AM_{n-1}}{M_n} \frac{3}{4} M_n \\
&= \frac{3}{4} AM_{n-1} > CM_{n-1},
\end{aligned}$$

and this contradicts our choice of f, proving the result. ∎

(4.2) Next we show that ρ_1 may be surjective.

4.2 Theorem. *There exists a sequence M for which every element of $l_1(M_n^{-1})$ has an extension in $E(1, M)$.*

Proof. It is enough to show that for M that grow sufficiently rapidly, each element of $l_1(M^{-1})$ has an extension in $D(I, M)$. This is because we may then multiply by a test function that is identically 1 near 0 and has support in I, and get an extension to $\mathbb{R}$. As long as M is an algebra sequence and grows rapidly enough to be nonquasianalytic, the test function may be chosen in $E(1, M)$, and hence the extension also belongs to $E(1, M)$.

Because of the nature of the space l_1 it is enough to find a sequence M, a constant C and a sequence of functions $(f_n)_{n=0}^{\infty}$ such that each f_n is in $D(I, M)$, $\|f_n\| \le C$, and $f_n^{(t)}(0) = M_n \delta_{t,n}$ for all t, n. We shall achieve this with constant $C = 3$ by making an inductive choice of the positive number M_n and the infinitely differentiable function f_n on I.

We begin by setting M_0=1, and take for f_0 the constant function 1. Having chosen M_j and f_j for $j < n$, we choose M_n large enough so that, for $j < n$, $\|f_j^{(n)}\|_I < 2^{-n} M_n$, and such that $M_n \ge \binom{n}{k} M_{n-k} M_k$ for all $k < n$. Now choose an infinitely differentiable function f_n such that $\|f_n^{(n)}\|_I = M_n$, $f_n^{(t)}(0) = M_n \delta_{t,n}$ for all t, and such that, for $j < n$, $\|f_n^{(j)}\|_I \le 2^{-j} M_j$. This can be done easily by making the support of $f_n^{(n)}$ narrow enough, since we are working only on the interval I. The inductive choice may now proceed.

It remains to show that each f_n is in $D(I, M)$, with norm at most 3. For $j = n$ we have $\|f_n^{(j)}\|_I = M_j$, while, for all other j,

$$\|f_n^{(j)}\|_I \le 2^{-j} M_j.$$

Thus

$$\sum_{j=0}^{\infty} (M_j^{-1} \|f_n^{(j)}\|_I) < 3,$$

which completes the proof. ∎

Remarks

1. The nature of the space l_1 gives us a continuous, linear extension operator. The above argument gives an extension operator of norm at most 3, but we can arrange for the norm to be as close to 1 as we like.

2. One particular sequence M_n for which ρ_1 is onto is the sequence 2^{2^n}. To modify the above argument to show this, we first note that there is a function $\phi \in E(1, \infty, (n!)^2)$ whose support is in I and such that $\phi^{(n)}(0) = \delta_{0,n}$. Let A be the norm of ϕ in $E(1, \infty, (n!)^2)$. Now, for $\epsilon \in (0, 1)$ and a non-negative integer k, we can consider the function $\psi_{k,\epsilon}$ in $D(I, (n!)^2)$ obtained by integrating the function $\phi(x/\epsilon)$ k times, i.e. $\psi_{k,\epsilon}^{(n)}(0) = \delta_{k,n}$ for all n, and $\psi_{k,\epsilon}^{(k)}(x) = \phi(x/\epsilon)$. For this crude choice of function, we have the following estimates for the derivatives. For $n < k$,

$$\|\psi_{k,\epsilon}^{(n)}\|_I \le A\epsilon.$$

For $n \geq k$ we have

$$\|\psi_{k,\epsilon}^{(n)}\|_I \leq A \frac{((n-k)!)^2}{\epsilon^{n-k}}.$$

Now, set $\epsilon_k = M_k^{-1} = 2^{-2^k}$, and set $f_k = M_k \psi_{k,\epsilon_k}$. We have $f_k^{(n)}(0) = M_k \delta_{k,n}$. For $n < k$ we have

$$\|f_k^{(n)}\|_I \leq A.$$

For $n \geq k$ we have

$$\|f_k^{(n)}\|_I \leq A M_k^{(n-k+1)} (n-k)!^2.$$

It follows that $f_k \in D(I, M)$ and that the norms of the functions f_k are bounded, as required.

(4.3) Consider compact sets K of the form

$$K = \{0\} \cup \bigcup_{n=0}^{\infty} [a_n, b_n]$$

where (a_n), (b_n) are sequences of positive numbers satisfying

$$b_n > a_n > b_{n+1}$$

and

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} b_n = 0.$$

We call these kind of sets *simple sets*. We shall denote the interval $[a_n, b_n]$ by I_n.

4.3 Theorem. *For any sequence M there is a compact simple set K and a function f on K such that $f \in D_1(K, 1, M)$, but f does not belong to $W(K, \infty, M)$, and hence f has no extension in $E(1, \infty, M)$. Thus f belongs to each $D_1(K, r, M)$ but has no extension in any $E(1, r, M)$.*

Proof. In fact, we show that for suitable sequences (a_n) and (b_n) the function f can be chosen to be constant on each interval I_n, with value c_n, say, and with $f(0) = 0$. For such a function to be infinitely differentiable on K it is necessary and sufficient that

$$\lim_{n \to \infty} \frac{c_n}{a_n} = 0.$$

All derivatives of positive order of f on K are then 0, and so $f \in D(K, r, M)$ for all $r \geq 1$. By Whitney's theorem, for such a function f to have an extension in $C^\infty(\mathbb{R}^d)$ it is necessary and sufficient that, for all k,

$$|f(x) - f(y)| = o(|x - y|)^k$$

on K.

Now let $(A_n)_{n=0}^{\infty}$ be a strictly increasing sequence of positive real numbers to be determined later, such that $\sum A_n^{-1}$ converges. Set

$$b_n = 2\sum_{k=n}^{\infty} \frac{1}{A_k},$$

$$a_n = \frac{1}{A_n} + 2\sum_{k=n+1}^{\infty} \frac{1}{A_k}$$

(the precise width of the intervals is not critical, but the distance between consecutive intervals is). Set

$$c_n = (-1)^n A_n^{-n/2}.$$

The function f obtained then satisfies the above conditions, because, for $x \in I_n$,

$$|f(x) - f(0)| \le |x - 0|^{n/2},$$

while for $x \in I_m, y \in I_n$ we have

$$|f(x) - f(y)| \le |x - y|^{m/2} + |x - y|^{n/2}.$$

Thus f is in $D_1(K, r, M)$ for all $r \ge 1$. To show that, for suitable choice of A_n, f does not belong to $W(K, \infty, M)$, observe that

$$|f(a_n) - T_{b_{n+1}}^{n-1} f(a_n)| = |c_n - c_{n+1}|,$$

so

$$\begin{aligned} \|f\|'_{K,M} &\ge {M_n}^{-1} n!(a_n - b_{n+1})^{-n} |c_n - c_{n+1}| \\ &\ge {M_n}^{-1} n! A_n^{\frac{n}{2}}. \end{aligned}$$

There is no difficulty in choosing A_n to satisfy the above conditions and also such that $n!A_n^{n/2}/M_n$ is unbounded, and so the result follows.

In particular, even for the sequence 2^{2^n} and this relatively simple class of compact set, functions in $D_1(K, 1, M)$ may not have extensions in the appropriate class.

(4.4) Next, we show that for any sequence M it is possible to extend from a point to some perfect set containing the point.

4.4 Theorem. *For each sequence M there is a simple compact set K and a continuous linear operator $S : l_1(M_n^{-1}) \longrightarrow D(K, M)$ satisfying*

$$(S(\alpha))^{(n)}(0) = \alpha_n.$$

Proof. Note that the space $D(K, M)$ will be incomplete, so that we must proceed with caution. We shall choose sequences a_n, b_n, with b_n decreasing, and then define S by

$$S(\alpha)(0) = \alpha_0$$

$$S(\alpha)|I_n = \sum_{k=0}^{n} \frac{\alpha_k}{k!} x^k,$$

so that $S(\alpha)$ is the appropriate Taylor polynomial of degree n on the interval I_n. With this definition, $S(\alpha)$ will be infinitely differentiable on K and $(S(\alpha))^{(n)})(0) = \alpha_n$ provided that, for all n,

$$\lim_{m\to\infty} \left\| \frac{1}{x} \left(\left(\sum_{k=0}^{m-n} \frac{\alpha_{n+k}}{k!} x^k \right) - \alpha_n - \alpha_{n+1} x \right) \right\|_{I_m} = 0.$$

For this, it is sufficient that, for all n,

$$\lim_{m\to\infty} \sum_{k=2}^{m-n} \left| \frac{\alpha_{n+k}}{k!} \right| b_m^{k-1} = 0.$$

This will be satisfied for all α in $l_1(M_n^{-1})$ if the sequence b_n is such that $b_{n+1} \le b_n$ and

$$b_n \le \min \left\{ \frac{1}{(n+1)}, \min \left\{ \frac{M_k^{-1}}{(n+1)} : 1 \le k \le n \right\} \right\},$$

which is a modest requirement. Now let $\alpha \in l_1(M_n^{-1})$ and set $f = S(\alpha)$. We estimate the norm of f. On the interval I_m we have $f^{(n)} = 0$ for all $n > m$, and, for $n \le m$,

$$\begin{aligned} |f^{(n)}(x)| &\le |\alpha_n| + \left| \sum_{k=1}^{m-n} \frac{\alpha_{n+k}}{k!} x^k \right| \\ &\le |\alpha_n| + \sum_{k=1}^{m-n} |\alpha_{n+k}| b_{n+k}^k. \end{aligned}$$

Thus, if b_n has the above properties, and in addition satisfies $b_{n+k} \le 2^{-n} M_{n+k}^{-1} M_n$ for all n, k, then

$$\|f^{(n)}\|_K \le |\alpha_n| + 2^{-n} M_n \|\alpha\|$$

and so $f \in D(K, M)$, and

$$\|f\| \le \|\alpha\| (1 + \sum_{n=0}^{\infty} 2^{-n}) = 3\|\alpha\|.$$

This proves the result, because we may easily choose a strictly decreasing sequence b_n with these properties and such that $b_n \to 0$. We may then choose any value for a_n between b_{n+1} and b_n. ∎

5. More approximation results in one and two dimensions.

(5.1) The results so far show that there is in general no way forward based on the use of an extension operator globally–defined on $D(K, M)$, although such an operator may exist in some cases. We proceed now to use densely–defined extension operators. For the application to approximation by rational functions and polynomials, these operators are just as good. We show that they exist at least in the totally-disconnected case and in the one–dimensional case.

We suppose throughout this section that M is a nonanalytic algebra sequence.

It is worth noting that even when $D(K, M)$ is incomplete, it is still true that if f_n is a Cauchy sequence in $D(K, M)$ and g is an infinitely differentiable function on a neighbourhood of K such that $f_n^{(k)}$ tends to $g^{(k)}$ uniformly on K for all k, then g is in $D(K, M)$ and f_n converges to g in $D(K, M)$.

We begin with some general remarks about polynomial and rational approximation in the case when $D(K, M)$ is not necessarily complete.

The function $r_a : z \mapsto 1/(z - a)$ belongs to $D(K, M)$ whenever $a \notin K$, because its derivatives are controlled by $k! \cdot \mathrm{dist}(z, K)^{-k}$. The map $a \mapsto r_a$ is a continuous function from $\mathbb{C} \sim K$ into $D(K, M)$.

Lemma 5.1. *The set of those $a \in \mathbb{C}$ for which r_a belongs to the closure of the polynomials in $D(K, M)$ is precisely the unbounded component of $\mathbb{C} \sim K$.*

Proof. This proof uses what is known as the usual Runge argument. We include it here for the convenience of the reader, and the comfort of those who are used to seeing it in the context of complete spaces. It does not depend on completeness of $D(K, M)$.

If a sequence $\{f_n\}$ converges in $D(K, M)$ norm, then it converges uniformly on K. Thus r_a is certainly not the $D(K, M)$–norm limit of a sequence of analytic polynomials unless a belongs to the unbounded component Ω of $\mathbb{C} \sim K$. It remains to see that r_a is such a limit if a does belong to Ω.

Let P denote the closure of $\mathbb{C}[z]$ in $D(K, M)$.

Fix $a \in \Omega$. When $|b|$ is large enough, the power series

$$r_b(z) = -\sum_{n=0}^{\infty} \frac{z^n}{b^{n+1}}$$

converges in $D(K, M)$ norm, so $r_b \in P$. Pick such a b, and choose a curve $\gamma : [0, 1] \to \Omega$ such that $\gamma(0) = a$ and $\gamma(1) = b$. Let E denote the set of $t \in [0, 1]$ such that $r_{\gamma(t)} \in P$. Then $1 \in E$, so E is nonempty. Evidently, E is closed. We claim that E is also open. This is a consequence of the fact that the set

$$\{c \in \mathbb{C} : r_c \in P\}$$

is open, and this follows from the $D(K, M)$–norm–convergent series representation

$$r_d = \sum_{n=0}^{\infty} (d - c)^n r_c^{n+1},$$

which holds whenever $|d-c|$ is smaller than $1/\|r_c\|_{D(K,M)}$. By the connectivity of $[0,1]$ it follows that $E=[0,1]$, hence $0\in E$, hence $r_a\in P$. ∎

It follows easily that each rational function having poles off K is a limit of polynomials in $D(K,M)$ if and only if $\mathbb{C}\sim K$ is connected.

If f is a function defined and holomorphic on some neighbourhood of K, then the restriction $f|K$ may be represented by a contour integral on a smooth contour lying in this neighbourhood and outside K. The Cauchy integral formulas for the derivatives imply that $|f^{(k)}|$ is bounded on K by $Ld^{-k-1}k!/2\pi$, where L is the length of the contour and d is the least distance from the contour to K. Thus $f\in D(K,M)$, since M is non–analytic. The continuity of the map $a\mapsto r_a$ implies that f may be approximated in $D(K,M)$ norm by (finite) Riemann sums of the form $\sum_j \lambda_j r_{a_j}$, with $\lambda_j\in\mathbb{C}$. Thus f is a limit of rational functions in $D(K,M)$ norm.

We employ the usual notation $\mathbb{C}[z]$ for the space of analytic polynomials, and $\mathbb{C}(z)$ for the ring of quotients. We regard both as function spaces.

Theorem 5.2. *Suppose that $K\subset\mathbb{C}$ is compact, perfect and totally–disconnected. Let M be a non–analytic algebra sequence. Then $\mathbb{C}[z]$ is a dense subset of $D(K,M)$.*

The proof depends on the following lemma. Before stating it, we recall that a set K has dimension zero at a point a (abbreviated $\dim_a K=0$) if each neighbourhood of a contains a neighbourhood whose boundary is disjoint from K.

Lemma 5.3. *Let $K\subset\mathbb{C}$ be compact and perfect and let M be any non–analytic algebra sequence. Let $f\in D(K,M)$ and let $\epsilon>0$. Then there exists $N\in\mathbb{N}$ such that, whenever $a\in K$ satisfies $\dim_a K=0$, then there exists $g\in\mathbb{C}[z]$ with degree at most N and a closed neighbourhood U of a having $K\cap \mathrm{bdy}U=\emptyset$, such that*

$$\frac{|(f-g)^{(k)}(x)|}{M_k}\le\frac{\epsilon}{2^{k+1}},\ \forall x\in U\cap K,\ \forall k\le N,$$

and

$$\sum_{k>N}\frac{\|f^{(k)}\|_K}{M_k}<\frac{\epsilon}{2}.$$

Proof. Pick $N\in\mathbb{N}$ such that

$$\sum_{N+1}^{\infty}\frac{\|f^{(k)}\|_K}{M_k}<\epsilon/2.$$

Let g be the N-th order Taylor polynomial of f about a. Observe that g is an *analytic* polynomial, hence $g|K\in D(K,M)$.

Pick $r>0$ such that $|x-a|<r$ and $x\in K$ imply that

$$\frac{|(f-g)^{(k)}(x)|}{M_k}\le\frac{\epsilon}{2^{k+1}},\ \forall k\le N.$$

Pick a neighbourhood U of a such that U is contained in the ball of radius r about a and $K\cap\mathrm{bdy}U=\emptyset$. This is possible since the dimension of K at a is zero. ∎

Proof of Theorem. Fix $f \in D(K,M)$ and $\epsilon > 0$. Choose N as in Lemma 5.3.

By compactness, we may choose points $a_1, \ldots, a_n$ in K and open neighbourhoods U_1, ..., U_n of these points (respectively) and polynomials g_1, ..., $g_n \in \mathbb{C}[z]$, such that K is covered by the U_j's, $K \cap \text{bdy} U_j = \emptyset$,

$$\frac{|(f - g_j)^{(k)}|}{M_k} \leq \frac{\epsilon}{2^{k+1}} \text{ on } U_j \cap K, \ \forall k \leq N,$$

and

$$\sum_{k>N} \frac{\|f^{(k)}\|_K}{M_k} < \frac{\epsilon}{2}.$$

By removing parts of some U_j, if need be, we may ensure that the U_j are pairwise–disjoint, without affecting the above properties. Define g on $\bigcup_j U_j$ by setting $g|U_j = g_j$. Then g is holomorphic on a neighbourhood of K, and hence is a $D(K,M)$ limit of polynomials.

For $k \leq N$,

$$\begin{aligned} \frac{\left\|(f-g)^{(k)}\right\|_K}{M_k} &= \max_j \frac{\|(f-g)^{(k)}\|_{K \cap U_j}}{M_k} \\ &\leq \frac{\epsilon}{2^{k+2}}. \end{aligned}$$

For $k > N$,

$$\frac{\|(f-g)^{(k)}\|_K}{M_k} = \frac{\|f^{(k)}\|_K}{M_k}.$$

Thus

$$\begin{aligned} &\sum_{k=0}^{\infty} \frac{\|(f-g)^{(k)}\|_K}{M_k} \\ &\leq \sum_{k \leq N} \frac{\epsilon}{2^{k+2}} + \sum_{k>N} \frac{\|f^{(k)}\|_K}{M_k} \\ &\leq \sum_{k=0}^{\infty} \frac{\epsilon}{2^{k+2}} + \sum_{k>N} \frac{\|f^{(k)}\|_K}{M_k} < \epsilon. \end{aligned}$$

■

Remark. Note that under the hypotheses of the theorem, this proof shows that $E(1,M) \cap D(K,M)$ is dense in $D(K,M)$, provided M is non–quasianalytic.

In one dimension, we can deal with general compact sets by using a variation of the same idea. The components of a one–dimensional compact set are either points or closed intervals of positive length. Point components can be handled using Lemma 5.3. The following lemma covers the other case.

Lemma 5.4. *Let M be a non–analytic algebra sequence. Let K be a compact perfect subset of $\mathbb{R}$. Let $a \in K$ and let the connected component of a in K be be a closed interval J of positive length. Let $f \in D(K,M)$ and $\epsilon > 0$. Then there exist $N \in \mathbb{N}$, an open interval U containing J, and a function $g \in D(\text{clos}U, M)$, such that $K \cap \text{bdy}U = \emptyset$,*

$$\sum_{k>N} \frac{\|f^{(k)}\|_K}{M_k} < \frac{\epsilon}{4},$$

$$\|g^{(k)}\|_{K\cap U} \leq \|f^{(k)}\|_K, \ \forall k,$$

and

$$\frac{\|g^{(k)} - f^{(k)}\|_{K\cap U}}{M_k} \leq \frac{\epsilon}{2^{k+2}}, \ \forall k \leq N.$$

Proof. Let m be the midpoint of J. Without loss of generality we may take $m = 0$.

Pick $N \in \mathbb{N}$ such that

$$\sum_{k>N} \frac{\|f^{(k)}\|_K}{M_k} < \epsilon/4.$$

Define

$$g_\alpha(x) = f(x/\alpha), \ \forall x \in \alpha J, \ \forall \alpha > 1.$$

We have

$$g_\alpha^{(k)}(x) = \alpha^{-k} f^{(k)}(x/\alpha), \ \forall x \in \alpha J,$$

so

$$\|g_\alpha^{(k)}\|_{\alpha J} \leq \|f^{(k)}\|_J$$

for each k. Now for each k, $g_\alpha^{(k)} \to f^{(k)}$ uniformly on J, so we may choose $\alpha > 1$ such that

$$\|g_\alpha^{(k)} - f^{(k)}\|_J < \frac{\epsilon M_k}{2^{k+2}},$$

for each $k \leq N$. By continuity of $f^{(k)}$ on K, we may then choose an open interval U containing J and contained in αJ, such that the boundary of U does not meet K and

$$\|g_\alpha^{(k)} - f^{(k)}\|_{U\cap K} < \frac{\epsilon M_k}{2^{k+2}},$$

for each $k \leq N$. Taking $g = g_\alpha$, we are done. ∎

Theorem 5.5. *Suppose $K \subset \mathbb{R}$ is compact and perfect and M is a non–analytic algebra sequence. Then the analytic polynomials are dense in $D(K,M)$.*

Proof. Fix $f \in D(K,M)$ and $\epsilon > 0$.

For each $a \in K$, we may apply either Lemma 5.3 or Lemma 5.4 to obtain an open interval U_a, containing the connected component of a in K, and a function $g_a \in D(\text{clos}U_a, M)$, and a number $N_a \in \mathbb{N}$, such that $K \cap \text{bdy}U_a$ is empty,

$$\sum_{k>N_a} \frac{\|f^{(k)}\|_K}{M_k} < \frac{\epsilon}{4},$$

$$\|g_a^{(k)}\|_{K\cap U_a} \le \|f^{(k)}\|_K, \ \forall k > N_a,$$

and

$$\frac{\|g_a^{(k)} - f^{(k)}\|_{K\cap U_a}}{M_k} \le \frac{\epsilon}{2^{k+2}}, \ \forall k \le N_a.$$

By compactness we may select $a_1, \ldots, a_n$ such that $K \subset \bigcup U_{a_j}$. Removing parts of some U_{a_j}, if need be, we may assume that the closU_j are pairwise–disjoint. We abbreviate $K_j = \text{clos}U_j$, and we set $L = \min_j N_{a_j}$.

By the polynomial approximation theorem for intervals [O1], we may choose polynomials $h_j \in \mathbb{C}[z]$ with

$$\|g_j - h_j\|_{D(K_j,M)} < \frac{\epsilon}{4n}.$$

Define

$$g(x) = g_j(x), \ \forall x \in K_j \text{ and}$$
$$h(x+iy) = h_j(x+iy), \ \forall x \in K_j, \ \forall y \in \mathbb{R}.$$

Then h is holomorphic on a neighbourhood of K and

$$\begin{aligned}
\|g - f\|_{D(K,M)} &\le \sum_{k\le L} \frac{\|g^{(k)} - f^{(k)}\|_K}{M_k} + \sum_{k>L} \frac{\|g^{(k)}\|_K}{M_k} + \sum_{k>L} \frac{\|f^{(k)}\|_K}{M_k} \\
&\le \sum_{k\le L} \frac{\epsilon}{2^{k+2}} + 2\sum_{k>L} \frac{\|f^{(k)}\|_K}{M_k} \\
&\le \frac{3\epsilon}{4}. \\
\|h - f\|_{D(K,M)} &\le \|h - g\|_{D(K,M)} + \|g - f\|_{D(K,M)} \\
&\le \sum_{j=1}^{n} \|h_j - g_j\|_{D(K_j,M)} + \|g - f\|_{D(K,M)} \\
&\le n \cdot \frac{\epsilon}{4n} + \frac{3\epsilon}{4} = \epsilon.
\end{aligned}$$

■

This method can be used to deal with finite disjoint unions of the types of set so far dealt with. We formalise this remark:

Theorem 5.6. *Let M be a nonanalytic algebra sequence. Suppose that K is a compact perfect subset of $\mathbb{C}$, and that K is a finite disjoint union of compact subsets L_n each of which is either totally disconnected, or a subset of a straight line segment, or a translate of a radially–self–absorbing set. Then $\mathbb{C}[z]$ is dense in $D(K, M)$.*

Proof. Observe that the complement of K is necessarily connected, so all we have to show is that the space of functions holomorphic on a neighbourhood of K is dense in $D(K, M)$.

Let $f \in D(K, M)$ and $\epsilon > 0$. For each of the compact subsets L_n we may use either Corollary 3.2, Theorem 5.2 or Theorem 5.5 to obtain a polynomial g_n such that the $D(L_n, M)$ norm of $f - g_n$ is less than $\frac{\epsilon}{2^n}$. We may also choose open sets U_n whose closures are disjoint such that $L_n \subseteq U_n$, and the closure of U_n misses $K \sim L_n$. Now define g on the union of the open sets U_n such that the restriction of g to U_n is g_n. Then g is holomorphic on a neighbourhood of K, and, adding our estimates for the norms of $f - g_n$, we see that the $D(K, M)$ norm of $f - g$ is less than ϵ. The result follows. ∎

Acknowledgement. We are grateful to the referee for some valuable criticism.

References.

[BA] T. Bang, 'The theory of metric spaces applied to infinitely– differentiable functions', *Math. Scand.*, 1 (1953), 137-152.
[BR] J. Bruna, 'An extension theorem of Whitney type for non quasi-analytic classes of functions', *J. London Math. Soc.* (2), 22 (1980), 495-505.
[CM] T. Carleman, *Les fonctions quasi analytiques*, Gauthier-Villars, Paris, 1926.
[DD] H.G. Dales and A.M.Davie., 'Quasianalytic Banach function algebras', *J. Funct. Anal.* 13 (1973), 28-50.
[DM] H.G. Dales and J.P. McClure, 'Completion of normed algebras of polynomials. *J. Austral. Math. Soc.* (A) 20 (1975), 504-510.
[H]T. Honari, 'Polynomial and rational approximation in Banach function algebras of infinitely differentiable functions on closed discs and annuli', *Journal of Science*, University of Teacher Education, Tehran, to appear.
[K] B.I. Korenblum, 'Quasianalytic classes of functions in a circle', *Soviet Math. Dokl.* 6 (1965), 1155-1158.
[M] B.S. Mityagin, 'An indefinitely differentiable function with the values of its derivatives given at a point', *Soviet Math.*, 2 (1961), 594-597.
[O1] A.G. O'Farrell, 'Polynomial approximation of smooth functions', *J. London Math. Soc.* (2) 28 (1983), 496-506.
[O2] —, 'T–invariance', *Proceedings Roy. Ir. Acad.*, 92A (1992) 185–203.
[R] W. Rudin, Real and Complex Analysis, 3rd edition, McGraw–Hill, 1986
[W] G. Wahde, 'Interpolation in non-quasianalytic classes of infinitely differentiable functions', *Math. Scand.*, 20 (1967), 19-31.

H. Lande, A.G. O'Farrell, Mathematics Department, Maynooth College, County Kildare, Ireland.

J.F. Feinstein, Mathematics Department, Nottingham University, Nottingham, England.
Draft 9., 12–9–94