# Conjugacy for Orientation-preserving Diffeomorphisms that fix only the Ends of an Interval

Anthony G. O'Farrell and Maria Roginskaya

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## Abstract

Given a group $G$, the conjugacy problem in $G$ is the problem of giving an effective procedure for determining whether or not two given elements $f, g \in G$ are conjugate. This paper is about the conjugacy problem in the group $G = \mathrm{D}(I)$ of all orientation-preserving diffeomorphisms of an interval $I \subset \mathbb{R}$. The kernel of this problem is the special case in which the diffeomorphisms $f, g \in \mathrm{D}(I)$ are fixed-point-free on the interior $J$ of $I$. The problem is trivial if $I$ is open. We present an effective way to approach it when $I$ is half-open or compact.


## 1 Introduction

### 1.1 Objectives

We are going to work with orientation-preserving diffeomorphisms defined just on various intervals (open, closed, or half-open, bounded or unbounded). We denote the group of orientation-preserving diffeomorphisms of the interval $I \subset \mathbb{R}$ by $\mathrm{D}(I)$. If an endpoint $c$ belongs to $I$, then statements about derivatives at $c$ should be interpreted as referring to one-sided derivatives.

The conjugacy problem in the group of all diffeomorphisms of $I$ can be reduced to the conjugacy problem in the subgroup $\mathrm{D}(I)$ [OR], and this in turn can be reduced to the special case of conjugacy on an interval between maps that fix only the end(s) of the interval. The present paper focusses on this special case. We shall describe a reasonably practical way to determine conjugacy.

There has been much work on this problem. Important steps are the work of Sternberg, Takens, Sergeraert, Robbin, Mather (private communication), Young, and Kopell, among others. There is a useful summary survey of progess up to 1995 by Ahern and Rosay [AR]. See also references [S], [T], [SE], [RO], [M], [KCG, Chapter 8], [KH, Chapter 2], [MS], [Y], [B], [ALY].

## 1.2 Notation

fix($f$): the set of fixed points of $f$.

$f^{\circ n}$: the $n$-th iterate of $f$ (i.e the $n$-th power in the group $\mathrm{D}(I)$). We also use it for negative $n = -m$, to denote the $m$-th iterate of the inverse function $f^{\circ -1}$. The notation $f^{\circ 0}$. denotes the identity map $\mathbb{1}$.

$g^h$: $h^{\circ -1} \circ g \circ h$, whenever $g, h \in \mathrm{D}(I)$. We say that $h$ *conjugates* $f$ *to* $g$ if $f = g^h$.

## 1.3 Open Intervals

For open intervals, there are just two conjugacy classes of fixed-point-free maps:

**Proposition 1.1 (Sternberg)** *Suppose $I$ is an open interval and $f$ and $g$ are fixed-point-free elements of* $\mathrm{D}(I)$. *Then $f$ and $g$ are conjugate in* $\mathrm{D}(I)$ *if and only if their graphs lie on the same side of the diagonal.* ∎

In Sections 2-4 we study conjugacy on half-open intervals. This is the main meat of the paper.

The results are summarised in Section 2. In Subsection 2.6 we use the results about half-open intervals to address conjugacy in $\mathrm{D}(I)$, for compact intervals $I$, for maps that are fixed-point-free on the interior $J$ of $I$. We go on to give a useful necessary condition (the "shape condition"), and to discuss flowability for the compact case.

## 1.4 Remarks

Typically, if $f$ and $g$ are conjugate diffeomorphisms, then the family of diffeomorphisms $\phi$ such that $f = \phi^{\circ -1} \circ g \circ \phi$ is a left coset of the centraliser $C_f$ of $f$ (and a right coset of $C_g$). The problem of describing $C_f$ is a special conjugacy problem — which maps conjugate $f$ to itself?

There may be a great many conjugacies between two given conjugate diffeomorphisms. In the open-interval case, the centraliser of a fixed-point-free diffeomorphism is very large, and is not abelian.

Kopell [K] showed that when $I$ has one of its endpoints as a member, then the centraliser of $f$ must be quite small — it is a subgroup of a one-parameter abelian group, and it may consist just of the iterates of $f$. An example was given by Sergeraert [SE]; possibly this behaviour is "generic". Kopell [K] showed that it is generic when $I$ is a compact interval. Thus there is a connection between our subject and the question of when $f$ embeds in a flow. For this, see also [SE]. We shall make one or two remarks about imbeddings in flows as we go along (cf. Subsection 2.8 and Proposition 5.3).

# 2 Preliminaries, History and Main Result

## 2.1 The semigroups $S_\pm$

Consider the conjugation of diffeomorphisms of a half-open interval $I$, assuming that they are fixed-point free on the interior $J$ of the interval. There is no loss in generality in considering just the case $I = [0, +\infty)$, so we do that.

Consider $f, g \in \mathrm{D}([0,\infty))$, fixed-point-free on $(0,\infty)$. Under what circumstances does there exist an $h \in \mathrm{D}([0,\infty))$ with $f = g^h$?

The set of all $f \in \mathrm{D}([0,\infty))$, that fix only 0 is the disjoint union of the two subsets

$$S_+ = \{f : f(x) > x \text{ on } (0,\infty)\}$$
$$S_- = \{f : f(x) < x \text{ on } (0,\infty)\}$$

each of which is a sub-semigroup of $\mathrm{D}([0,\infty))$. Each of these semigroups is preserved by conjugacy, i.e. is a union of conjugacy classes. Thus, for $f$ to be conjugated to $g$ it is necessary that *they belong to the same semigroup, $S_+$ or $S_-$.*

Note that $f \in S_+$ is equivalent to $f^{\circ -1} \in S_-$, so that to characterize conjugacy it suffices to consider $f \in S_-$.

*Unless the contrary is indicated, we assume that both $f$ and $g$ belong to $S_-$ for the remainder of this section.*

## 2.2 The Hyperbolic Case

If $f = g^h$, then $g'(0) = f'(0)$, i.e. $f$ and $g$ have the same "multipliers" at 0. If the multiplier at 0 is not 1 (i.e. 0 is a *hyperbolic* fixed point), then Sternberg [S] identified the multiplier as the sole conjugacy invariant, and deduced that the centraliser of such a hyperbolic element is a one-parameter group.

## 2.3 Taylor Series

In general, there is a more elaborate necessary condition involving higher derivatives, best expressed in terms of Taylor series: Let $T_0 f$ denote the truncated Taylor series of $f$ about 0:

$$T_0 f = \sum_{n=1}^{\infty} \frac{f^{(n)}(0)}{n!} X^n$$

(regarded as a formal power series in an indeterminate $X$). One then has that $f = g^h$ implies

$$T_0 f = (T_0 h)^{\circ -1} \circ (T_0 g) \circ (T_0 h),$$

where $\circ$ denotes the formal composition, and $p^{\circ -1}$ denotes the formal compositional inverse. Thus $T_0 f$ and $T_0 g$ are conjugate in the group of formally-invertible series. We call this **Condition (T).** There is a straightforward algorithm for checking whether or not two formal power series are formally conjugate. The algorithm involves reducing them to normal forms [Ka, L, OF2].

If the multiplier is 1, but $f$ is not "infinitesimally tangent to the identity" (i.e. $T_0 f \neq X$ — we find it less of a mouthful to express this condition as "$f - x$ is not flat at 0"), then Takens [T, Theorem2] identified the conjugacy class of the Taylor series $T_0 f$ as the sole conjugacy invariant (— cf. also [AR] for another proof).

If $f - x$ is flat at 0, Condition (T) just says that $g - x$ is also flat at 0. This is not enough.

## 2.4 Example

**Proposition 2.1** *Let $f(x) = x - e^{-\frac{1}{x}}$ and $g(x) = x - e^{-\frac{1}{x^2}}$. The functions $f$ and $g$ are not conjugate.*

**Proof.** Suppose $h(x) = ax + bx^2 + \dots$ is a conjugation. Then it maps the interval $[\frac{x}{2}, x]$ to the interval $[\frac{ax}{2} + o(x), ax + o(x)]$. For small positive $x$, the first interval has $\mathrm{O}(x \exp(1/x))$ iterates of $x$ under $f$, whereas the second has $\mathrm{O}(x \exp(1/a^2 x^2))$ iterations of $h(x)$, a much greater number. But the conjugacy condition requires that the two intervals contain equal numbers of iterates of $x$ and $h(x)$, respectively. ∎

So we need another idea, in order to deal with two general elements $f, g \in S_-$.

If you think about it, the main meat of the conjugacy problem on $[0, +\infty)$ involves the functions with $f - x$ flat at 0. There is no hope of tackling the conjugacy problem for such functions by reducing to explicit normal forms. Neither is it possible to reduce it to the temptingly straightforward task of comparing vectorfields whose exponentials are the given functions, for the simple reason that the exponential map is not surjective. *The only way to come at it is to take two functions and compare them directly with one another, rather than with some collection of templates.* What is needed is a "practical" decision-procedure for determining conjugacy.

We find such a procedure by using an infinite product, and a differential equation.

## 2.5 The Product

For $x > 0$ and $\xi > 0$, let

$$H_1(x, \xi) = H_1(f, g; x, \xi) = \prod_{n=0}^{\infty} \frac{f'(f^{\circ n}(x))}{g'(g^{\circ n}(\xi))}. \tag{1}$$

We say that $f$ and $g$ satisfy **Condition (P)** if there exist $x > 0$ and $\xi > 0$ such that the product $H_1(x, \xi)$ converges.

The product $H_1(x, \xi)$ appears already in Sternberg's paper [S], in the special case $g(x) = \lambda x$, and in Kopell's paper [K] in the case $f = g$. We have not seen it used in the literature for general $f$ and $g$.

We shall show (Corollary 3.4) that if Condition (P) holds, then $H_1(x,\xi)$ exists for all $x>0$ and $\xi>0$, and (Lemma 4.1) is infinitely-differentiable and positive. We may then consider the three-parameter initial-value problem

$$D_1(a,\alpha,\lambda): \qquad \begin{cases} \dfrac{d\phi}{dx} &= H_1(x,\phi(x))\lambda,\ \forall x>0, \\ \phi(a) &= \alpha \end{cases} \tag{2}$$

depending on $\lambda>0$, $a>0$ and $\alpha>0$. We shall show that for each given $a>0$ and $\alpha>0$, there exists (Lemma 4.10) exactly one $\lambda>0$ for which the (unique) solution $\phi=\Phi_+(a,\alpha)$ to problem $D_1(a,\alpha,\lambda)$ has $f(a)=g^\phi(a)$, and (Lemma 4.8) that this $\phi$ conjugates $f$ to $g$ in $\mathrm{D}((0,+\infty))$, and (Lemma 4.9) extends in $C^1([0,+\infty))$. This means that, subject to Condition (P), there is a 1-parameter family of $C^1$ conjugations from $f$ to $g$ on $[0,+\infty)$[1]. This immediately gives us a result about $C^\infty$ conjugacy on $[0,+\infty)$:

**Theorem 2.2 (Main Theorem)** *Let $f,g\in S_-$. Then $f$ is conjugate to $g$ in $\mathrm{D}([0,+\infty)$ if and only if Condition (P) holds and there exists some $a>0$ and $\alpha>0$ for which $\Phi_+(a,\alpha)$ is $C^\infty$ at 0.*

The value of this result is that it narrows the search for a conjugating map $\phi$ to the 1-parameter family of solutions of an explicit ordinary differential equation.

We repeat (for emphasis) the fact already noted (cf. sections (3.1) and (3.2)) that when $f-x$ is not flat at 0, then Condition (T) implies $f$ is conjugate to $g$. The theorem is interesting when $f-x$ is flat at 0.

For a general half-open interval $I=[d,c)$ or $I=(c,d]$, we take $J=\mathrm{int}(I)$ and define $S_-$ as the semigroup of diffeomorphisms $f\in\mathrm{D}(I)$ which iterate all points of $J$ towards the endpoint $d$, and $S_+$ as the semigroup of those that iterate all points of $J$ towards $c$. In order to adapt the above results about $f,g\in S_-$ to the interval $J\cup\{d\}$, one should replace $(0,+\infty)$ by $J$, and 0 by $d$. Then, for $f,g\in S_-$, the product condition (P) takes precisely the same form (1), and the differential equation also, except that its domain is the interior $J$. The theorem yields, by conjugating $I$ to $[0,+\infty)$, a precisely similar result for $f,g\in S_-$ on $I$.

For $f,g\in S_+$, one applies the theorem to $f^{\circ-1}$ and $g^{\circ-1}$, which lie in $S_-$. Unwinding the definitions, we see that Condition (P) for elements of $S_+$ involves the infinite product

$$H_2(x,\xi)=\prod_{n=1}^{\infty}\frac{g'(g^{\circ-n}(\xi))}{f'(f^{\circ-n}(x))}, \tag{3}$$

and the differential equation takes the form:

$$D_2(a,\alpha,\mu): \qquad \begin{cases} \dfrac{d\phi}{dx} &= H_2(x,\phi(x))\mu, \\ \phi(a) &= \alpha. \end{cases} \tag{4}$$

[1] $\{\Phi_+(a,\alpha): a>0, \alpha>0\}$ is a 1-parameter family, because

$$\Phi_+(a,\alpha)=\Phi_+(b,\Phi_+(a,\alpha)(b))$$

for each $b>0$.

## 2.6 Compact Intervals

For a compact interval $I = [c, d]$, with nonempty interior $J$, we define $S_-(I)$ as the semigroup of homeomorphisms that iterate each element of $J$ towards $d$.

In order that two given $f, g \in S_-$ be conjugate in $\mathrm{D}([c, d]$, it is necessary that they be conjugate in $\mathrm{D}([c, d))$ and in $\mathrm{D}((c, d]))$. Thus the Main Theorem applies, and tells us that the two-sided product

$$H(x,\xi) = H_1(x,\xi) \cdot H_2(x,\xi)^{-1} = \prod_{n=-\infty}^{\infty} \frac{f'(f^{\circ n}(x))}{g'(g^{\circ n}(\xi))}$$

must converge for some (or equivalently all) $x, \xi \in J$. This is the appropriate version of Condition (P), for compact intervals.

Assuming Condition (P), we may form two initial-value problems, corresponding to equations (2) and (4). Given $a \in J$ and $\alpha \in J$, there are unique $\lambda$ and $\mu$, repectively, such that the solutions $\Phi_+(a, \alpha)$ and $\Phi_-(a, \alpha)$, respectively, to these equations conjugate $f$ to $g$ on $J$ and have $C^1$ extensions to $(c, d]$ and $[c, d)$, respectively. We may then formulate a solution to the conjugacy problem, as follows:

**Theorem 2.3** *Let $I$ be a compact interval and let $f, g \in \mathrm{D}(I)$, both fixed-point-free on $J$, both in $S_-$. Then the following conditions are equivalent:*
*(1) $f$ is conjugate to $g$ in $\mathrm{D}(I)$;*
*(2) The product $H(x, \xi)$ converges for some (and hence for all) $x > 0$ and $\xi > 0$, and there exists some $a > 0$ and $\alpha > 0$ such that the solution $\Phi_+(a, \alpha)$ extends $C^\infty$ to both ends of $I$;*
*(3) There exist $a > 0$ and $\alpha > 0$ such that $H(a, \alpha)$ converges, and $\Phi_+(a, \alpha) = \Phi_-(a, \alpha)$ extends in $\mathrm{D}(I)$.* ∎

## 2.7 Shape

It is worth noting a necessary condition (the "shape" condition) that is easier to check in the compact case. This will often suffice to show two maps are not conjugate.

First we define

$$F_a(x) = H(f, f; x, a) = \prod_{n=-\infty}^{\infty} \frac{f'(f^{\circ n}(x))}{f'(f^{\circ n}(a))},$$

and

$$G_\alpha(\xi) = H(g, g; \xi, \alpha) = \prod_{n=-\infty}^{\infty} \frac{g'(g^{\circ n}(\xi))}{g'(g^{\circ n}(\alpha))}$$

whenever $x, \xi, a, \alpha \in J$. Note that

$$H(x,\xi) \cdot G_\alpha(\xi) = F_a(x) \cdot H(a, \alpha), \tag{5}$$

whenever all the terms make sense.

**Proposition 2.4** *Suppose $f, g, h \in \mathrm{D}(I)$, $f$ is fixed-point-free on $J$, and $f = g^h$. Then $H(x, h(x))$ is constant on $J$. Thus, given any $a, \alpha \in J$, there is some $\kappa > 0$ such that*

$$F_a(x) = \kappa G_\alpha(h(x)), \ \forall x \in J.$$

This is proved in Section 5. This means that the graphs of each $F_a$ and of each $G_\alpha$ have the same "shape". If they are not monotone, then the relative diffeomorphism class of the critical set and the pattern of maxima and minima must be the same for both functions. The pattern for $F_a$ is determined by the pattern on the segment $I_a = [a, f(a)]$, because it repeats itself on successive images of $I_a$ under $f$. Similarly, the pattern for $G_\alpha$ is determined by the pattern on $[\alpha, g(\alpha)]$. Apart from this quasiperiodic feature, the patterns can be pretty complicated.

## 2.8 Flowability

We note an application to existence of a smooth flow (see below) on a compact interval, for which $f$ is the time-1 step.

By a *flow* on a compact interval $I$, we mean a continuous homomorphism $t \mapsto \Phi^t$ from the additive topological group $(\mathbb{R}, +)$ into $D(I)$, endowed with its usual topology (the topology of simultaneous convergence of functions and their inverses, uniformly on $I$).

We say that $f \in D(I)$ is *flowable* if there exists a flow $\Phi^t$, with $f = \Phi^1$ (i.e. $f$ is the "time 1" map of the flow $(\Phi^t)_{t\in\mathbb{R}}$.

**Proposition 2.5** *Suppose that $f \in \mathrm{D}(I)$, is fixed-point-free on $J$ and $f$ is flowable. Then for each $a \in J$, $F_a$ is either strictly monotone on $J$, or constant on $J$.*

**Proof.** Suppose that $F_a$ is neither strictly monotone on $J$ nor constant on $J$. Each conjugacy if $f$ to itself must permute the maximal open intervals of strict monotonicity of $F_a$. Since $F_a$ is smooth and not strictly monotone or constant, there exist at least two such intervals, and since the pattern repeats, there are in fact infinitely many. But the number is countable, since they are pairwise disjoint open sets, and conjugacy must permute the countable set of endpoints of these intervals of monotonicity, and is determined uniquely by the image of one endpoint. Hence the centralizer of $f$ is a countable group, not the image of a flow. ∎

We can do better when the graph of $f$ is tangent to the diagonal at the ends of $I$:

**Corollary 2.6** *Suppose $f \in \mathrm{D}(I)$ is fixed-point free on $J$ and is flowable. Then the following are equivalent:*

1. $f'(c) = f'(d)$;

*2.* $f'(c) = f'(d) = 1$;

*3.* $F_a$ *is constant on* $J$*, for each (or any one)* $a \in J$.

**Proof.** The implication (1) $\Rightarrow$ (2) follows from the fact that 1 is always trapped between $f'(c)$ and $f'(d)$.

Next, note that we have the formula

$$F_a(f(x)) = F_a(x)\frac{f'(d)}{f'(c)}, \tag{6}$$

whenever $a, x \in J$.

Suppose (2) holds. Fix $a \in J$. The formula 6 implies that $F_a(f(x)) = F_a(x)$ for all $x \in J$. Since $f$ is flowable, Proposition 2.5 tells us that $F_a$ is constant on each interval $[f(x), x]$. But for any fixed $x_0 = x$, the iterates $x_n = f^{\circ n}(x)$ converge monotonically to one end of $J$ as $n \uparrow +\infty$, and monotonically to the other end as $n \downarrow -\infty$, hence the intervals $[x_{n+1}, x_n]$ pave $J$, and, since $F_a$ is constant on each, it is constant on the whole interval $J$. Thus (2) $\Rightarrow$ (3).

Finally, suppose (3) holds. Then equation 6, applied to any $x \in J$, yields $f'(d) = f'(c)$, since $F_a(x)$ never vanishes. ■

We note that these results depend only on the assumption that $f \in C^2(I)$.

### 2.9 Remark

In special cases, the conjugacy problem on a compact interval can be reduced to condition (T) at both ends, plus identity of a suitable modulus (a conjugacy invariant that is a diffeomorphism on some interval). See Robbins [RO], Afraimovitch Liu and Young [ALY], and Young [Y]. All these results are subsumed in an unpublished lemma of Mather, subsequently and independently found by Young, which covers the case in which the germs of $f$ at both ends of the interval are the exponentials of smooth vector fields, and for which the modulus is a double coset of the rotation group in the group of circle diffeomorphisms, and the conjugacy class of $f$ is determined by the smooth conjugacy classes of the two vectorfield germs and the modulus. We are very grateful to Professor Mather for making his account of this available to us.

## 3 Half-open Interval: the Product Condition

Fix arbitrary $f, g \in S_-$.

**Lemma 3.1** *Suppose* $f$ *and* $g$ *are conjugate in* $\mathrm{D}([0,\infty))$*. Then for any* $x$ *and some corresponding* $\xi$ *the product (1) converges.*

**Proof.** Pick $h \in \mathrm{D}([0,\infty))$ with $f = g^h$, and set $\xi = h(x)$. We observe that $h \circ f^{\circ n} = g^{\circ n} \circ h$, hence equating derivatives we get

$$h'(f^{\circ n}(x))\frac{df^{\circ n}}{dx}(x) = \frac{dg^{\circ n}}{d\xi}(\xi)h'(x),$$

hence

$$\prod_{j=0}^{n-1} \frac{f'(f^{\circ j}(x))}{g'(g^{\circ j}(\xi))} = \frac{h'(x)}{h'(f^{\circ n}(x))},$$

so the product converges to the limit $h'(x)/h'(0)$. ∎

The correspondence between $x$ and $\xi$, referred to in the lemma is not essential, for we have the following, which is due to Kopell [K]. (We give the proof for convenience.)

**Lemma 3.2** *Let $x, y \in [0,\infty)$ and denote $x_n = f^{\circ n}(x)$, $y_n = f^{\circ n}(y)$. Then the infinite product*

$$\prod_{n=0}^{\infty} \frac{f'(x_n)}{f'(y_n)} \tag{7}$$

*converges.*

**Proof.** By removing a finite number of terms from the product, we may assume that $y_0$ is between $x_1$ and $x_0$. The convergence of the product is equivalent to the convergence of the series of logarithms $\sum\limits_{n=0}^{\infty} \ln(\frac{f'(x_n)}{f'(y_n)})$, which in turn is equivalent to that of $\sum\limits_{n=0}^{\infty}\left(1 - \frac{f'(x_n)}{f'(y_n)}\right)$. Now

$$\left|\frac{f'(x_n) - f'(y_n)}{f'(y_n)}\right| \le \left(\frac{\sup|f''|}{\inf|f'|}\right) \cdot |x_n - y_n|$$

(where the sup and inf are taken on $[0,x]$; note that the inf is positive since $f$ is a diffeomorphism), and so the convergence follows from $\sum\limits_{n=0}^{\infty} |x_n - y_n| \le |x_0|$, which holds because the intervals from $x_n$ to $y_n$ are pairwise-disjoint subintervals of that from 0 to $x_0$. ∎

**Corollary 3.3** *(1) In case $x_1 < y < x_0$, and $T_0 f = X + bX^{p+1} + \dots$ for some $p \in \mathbb{N}$, the product (7) is $1 + O(x^p)$ as $x \downarrow 0$, uniformly for $y$ between $x_1$ and $x_0$. (2) In case $T_0 f = X$ the product is $1 + o(x^n)$, for any $n$.*

**Proof.** (1) Just use the estimate $f''(x) = \mathrm{O}(x^{p-1})$.
(2) follows. ∎

**Corollary 3.4** *If the product (1) converges for some $x, \xi > 0$, then it converges for any choice of $x, \xi > 0$.* ∎

**Corollary 3.5** *Suppose $f$ and $g$ are conjugate in* $\mathrm{D}([0,\infty))$. *Then for any $x > 0$ and $\xi > 0$ the product (1) converges.* ■

**Corollary 3.6** *The convergence or divergence of the product (1) is not affected if the functions $f$ and $g$ are replaced by conjugates.* ■

It follows from the results of Sternberg [S] and Takens [T], mentioned already, that Condition (P) is actually a consequence of Condition (T) in the non-flat cases:

**Proposition 3.7** *(1) If $f'(0) \neq 1$ or $g'(0) \neq 1$, then Condition (P) is equivalent to $f'(0) = g'(0)$.*
*(2) If $f$ and $g$ have conjugate non-identity Taylor series, then Condition (P) is satisfied.* ■

Part (1) in fact is easy to prove, once observed. As for Part (2), by replacing $g$ with a conjugate which has the same Taylor series as $f$ we reduce to the case in which $f$ and $g$ have coincident Taylor series. The result then follows from the next, more general lemma, which we will use later.

**Lemma 3.8** *Let $T_0(f) = T_0(g) = X + bX^{p+1} + \cdots \pmod{X^{2p+1}}$, where $p \in \mathbb{N}$ and $b \neq 0$. Then*

$$\prod_{n=0}^{\infty} \frac{f'(f^{\circ n}(x))}{g'(g^{\circ n}(x))} = 1 + O(x^p).$$

**Proof.** Without loss in generality, we take $b < 0$, and write $c = -b$. We use $C$ for a positive constant that may differ at each occurrence. We may assume that the $x > 0$ under consideration are so small, that $|f(x) - x + cx^{p+1}| \le Cx^{p+2}$ and $|Cx| < \frac{1}{2}c$. This means that $cx_n^{p+1} - Cx_n^{p+2} < x_n - x_{n+1} < cx_n^{p+1} + Cx_n^{p+2}$. So, for $0 < \alpha < 1$ between $x$ and $\alpha x$ there are no more than

$$\frac{(1-\alpha)x}{(c(\alpha x)^{p+1} - Cx^{p+2})} = \frac{(1-\alpha)\alpha^{-(p+1)}}{cx^p(1-\alpha^{-(p+1)}Cx)}$$

and no fewer than

$$\frac{(1-\alpha)x}{(cx^{p+1} + Cx^{p+2})} = \frac{1-\alpha}{cx^p(1+Cx)}$$

points from the $f$-orbit of $x$.

Let us start by reformulating the claim: It is enough to prove that

$$\log\left(\prod_{n=0}^{\infty} \frac{f'(f^{\circ n}(x))}{g'(g^{\circ n}(x))}\right) = \sum_{n=0}^{\infty} \log\left(\frac{f'(f^{\circ n}(x))}{g'(g^{\circ n}(x))}\right) = \mathrm{O}(x^p).$$

As $f'(0) = g'(0) = 1$ and $|\log(t)| \sim |1-t|$ close to $t = 1$, it is enough to prove that

$$\sum_{n=0}^{\infty} |g'(g^{\circ n}(x)) - f'(f^{\circ n}(x))| = O(x^p).$$

Since $T_0 f = T_0 g$, we may also assume $x$ is so small that $|f'(x) - g'(x)| < Cx^{2p}$. We then observe that, since $|x_n - x_{n+1}| > (c/2)x_n^{p+1}$, we have the estimate

$$\sum_{k=0}^{\infty} x_k^{p+1} \leq \frac{2}{c} \sum_{k=0}^{\infty} |x_k - x_{k+1}| = \frac{2x_0}{c}. \tag{8}$$

As $|f'(x) - g'(x)| < Cx^{2p}$ for all $x$ in question, we have

$$\sum_{k=0}^{\infty} |f'(x_k) - g'(x_k)| \leq C \sum_{k=0}^{\infty} x_k^{2p} \leq C x_0^{p-1} \sum_{k=0}^{\infty} x_k^{p+1} = O(x_0^p),$$

and the estimate can be reduced to estimating $\sum |g'(x_n) - g'(g^{\circ n}(x))|$. Since $g'' = O(x^{p-1})$ we have $|g'(r) - g'(s)| \leq O(s^{p-1})|r - s|$ for $r < s$, and it remains to show that $\sum |f^{\circ n}(x) - g^{\circ n}(x)| < Cx$.

Let us now consider only points so close to the origin that $|f(x) - g(x)| < Cx^{2p+1}$. For those points we have the estimate

$$\begin{aligned} |f^{\circ n}(x) - g^{\circ n}(x)| &\leq |f^{\circ n}(x) - f(g^{\circ(n-1)}(x))| + |f(g^{\circ(n-1)})(x) - g^{\circ n}(x)| \\ &\leq M_x |f^{\circ(n-1)}(x) - g^{\circ(n-1)}(x)| + (g^{\circ(n-1)}(x))^{2p+1} \\ &\leq (M_x^n + \cdots + 1)x^{2p+1}, \end{aligned}$$

where $M_x$ is the maximum of $f'$ on the interval $[0, x]$, and thus can (for small $x$) be estimated from above by 1 (since $b < 0$). This gives us $|f^{\circ n}(x) - g^{\circ n}(x)| \leq nx^{2p+1}$.

Let us consider the first point in the orbit of $x$ with respect to $f$ which is less than $\alpha x$. Let it be $f^{\circ n_1}(x)$. Then by the observation at the beginning of the proof, for $\alpha > \frac{1}{2}$, $n_1 < (1 - \alpha)C/x^p$, where the constant depends only on the Taylor expansion. By the previous paragraph, for any $k \leq n_1$

$$|f^{\circ k}(x) - g^{\circ k}(x)| \leq \frac{(1 - \alpha)Cx^{2p+1}}{x^p} = (1 - \alpha)Cx^{p+1}.$$

As $|f(y) - y| > \frac{c}{2}y^{p+1}$, we see that for a choice of $\alpha < 1$ close enough to 1, $\frac{1}{2}(f^{\circ(k+1)}(x) + f^{\circ k}(x)) < g^{\circ k}(x) < \frac{1}{2}(f^{\circ k}(x) + f^{\circ(k-1)}(x))$. Thus the intervals $[f^{\circ k}(x), g^{\circ k}(x)]$ are disjoint, and $\sum_{k=0}^{n_1} |f^{\circ k}(x) - g^{\circ k}(x)| \leq (1 - \alpha)x + Cx^{p+1}$. On the other hand, in the particular case, $k = n_1$, if $g^{\circ(n_1+1)}(x) = x^{(1)}$, we have $\sum_{m=0}^{\infty} |f^{\circ(n_1+1+m)}(x) - f^{\circ m}(x^{(1)})| \leq \alpha x$, as the sum of lengths of disjoint intervals. This means that

$$\begin{aligned} &\sum_{n=0}^{\infty} |f^{\circ n}(x) - g^{\circ n}(x)| \\ &\leq \sum_{n=0}^{n_1} |f^{\circ n}(x) - g^{\circ n}(x)| + \sum_{m=0}^{\infty} |f^{\circ m}(x^{(1)}) - g^{\circ m}(x^{(1)})| + \end{aligned}$$

$$\sum_{m=0}^{\infty} |f^{\circ(n_1+1+m)}(x) - f^{\circ m}(x^{(1)})| + Cx^{p+1}$$

$$\leq (1-\alpha)x + \sum_{m=0}^{\infty} |f^{\circ m}(x^{(1)}) - g^{\circ m}(x^{(1)})| + \alpha x + Cx^{p+1} \leq$$

$$2x + \sum_{m=0}^{\infty} |f^{\circ m}(x^{(1)}) - g^{\circ m}(x^{(1)})|.$$

Using this argument inductively we deduce that

$$\sum_{n=0}^{\infty} |f^{\circ n}(x) - g^{\circ n}(x)| \leq 2x + 2x^{(1)} + \ldots \leq 2\sum_{j=0}^{\infty} \alpha^j x = Cx,$$

and we are done. ■

**Remark.** Notice, that for the particular case $p = 1$ this lemma says that the Condition (P) is satisfied for $f(x) = x + x^2$ and $g(x) = x + x^2 + x^3$. On the other hand, the Taylor series $X + X^2$ and $X + X^2 + X^3$ are not conjugate, which shows that the Condition (P) is strictly weaker than Condition (T) in the non-flat case .

We shall see shortly that condition (P) guarantees the existence of a $C^1$ diffeomorphism conjugating $f$ to $g$. Thus the existence of a $C^1$ conjugacy is strictly weaker than the existence of a $C^\infty$ conjugacy.

We mention here the observations of Young [Y]. He considered $C^2$ diffeomorphisms $f$ on $[0, +\infty)$ with $T_0 f = x + ax^2 \pmod{x^3}$, and with $a \neq 0$. A result of Szekeres (cf. [KCG, Theorem 8.4.5]) implies that all such $C^2$ diffeomorphisms (having $a$ of one sign) are $C^1$-conjugate. Young showed that they are in fact $C^2$-conjugate.

# 4 Half-open Interval: Sufficient Conditions

## 4.1 The Differential Equation

Suppose $f, g \in \mathrm{D}([0, +\infty))$ fix only 0, both belong to $S_-$ and satisfy condition (P).

We define

$$F_{1a}(x) = H_1(f, f; x, a) = \prod_{n=0}^{\infty} \frac{f'(f^{\circ n}(x))}{f'(f^{\circ n}(a))}$$

whenever $a, x > 0$. Note that

$$F_{1a}(x) = \lim_{n\uparrow\infty} \frac{(f^{\circ n})'(x)}{(f^{\circ n})'(a)}.$$

We define

$$G_{1\alpha}(\xi) = H_1(g, g; \xi, \alpha) = \prod_{n=0}^{\infty} \frac{g'(g^{\circ n}(\xi))}{g'(g^{\circ n}(\alpha))}$$

whenever $\alpha, \xi > 0$.

**Lemma 4.1** *Fix $a > 0$, $\alpha > 0$. The functions $x \mapsto F_{1a}(x)$ and $\xi \mapsto G_{1\alpha}(\xi)$ are infinitely-differentiable and positive on $(0 + \infty)$, and hence*

$$(x, \xi) \mapsto H_1(x, y) = H_1(a, \alpha)F_{1a}(x)/G_{1\alpha}(\xi)$$

*is infinitely-differentiable and positive on $(0, +\infty) \times (0, +\infty)$.*

**Proof.**

It suffices to show that $x \mapsto F_{1a}(x)$ is infinitely-differentiable on $(0, +\infty)$ for each $a > 0$. The argument for $\xi \mapsto G_{1\alpha}(\xi)$ is precisely analogous.

Fix $a \in (0, +\infty)$. Let $J_a$ denote the closed interval from 0 to $a$. Let $a_n = f^{\circ n}(a)$, for all $n \in \mathbb{Z}$. Let $I_a$ denote the closed interval from $a_1$ to $a$. Let

$$D_j = \max_{J_a} |f^{(j)}|, \forall j \in \mathbb{Z}.$$

(Note that $\min_{J_a} |f'| = (D_{-1})^{-1}$.)

For $x \in (0, +\infty)$, let $x_n = f^{\circ n}(x)$, for all $n \in \mathbb{Z}$. For ease of notation, we abbreviate $\frac{d}{dx} f^{\circ n}(x) = f'(x)f'(x_1) \ldots f'(x_{n-1})$ to $x'_n$, and similarly denote $\frac{d^k}{dx^k} f^{\circ n}(x)$ by $x_n^{(k)}$. We use $x''_n$ for $x^{(2)}$, etc.

Before continuing the proof, we pause to note a couple of lemmas that follow from Lemma 3.2.

In what follows, unless otherwise specified, we use $K$ to denote a constant that depends at most on $f$, and $a$, and that may be different at each occurence.

**Lemma 4.2**

$$K^{-1}|(f^{\circ n})'(a)| \leq |x'_n| \leq K|(f^{\circ n})'(a)|$$

*whenever $x \in I_a$.*

**Proof.**

$$\frac{(f^{\circ n})'(a)}{(f^{\circ n})'(x)} = \prod_{j=0}^{n-1} \frac{f'(a_j)}{f'(x_j)},$$

so the result follows from the uniform convergence of $\prod_{j=0}^{\infty} \frac{f'(a_j)}{f'(x_j)}$, for $x \in I_a$. ■

**Lemma 4.3**

$$|x'_n| \leq K \left| \frac{x_{n+1} - x_n}{x_1 - x_0} \right|$$

*whenever $x \in I_a$.*

**Proof.** By the Law of the Mean,

$$\frac{x_{n+1} - x_n}{x_1 - x_0} = (f^{\circ n})'(y)$$

for some $y$ between $x$ and $x_1$, so the result follows from a few applications of the previous lemma. ∎

**Lemma 4.4** *$|x_1 - x| \geq K|a_1 - a|$, for all $x \in I_a$.*

**Proof.** For $x \in I_a$, $f(a) \leq x \leq a$, so $f(x) \leq f(a) \leq x$, so $|f(x) - x| = |f(x) - f(a)| + |x - f(a)| \geq (D_{-1})^{-1}|x-a| + |x-a_1| \geq \min\{1, (D_{-1})^{-1}\}|a-a_1|$. ∎

**Proof of Lemma 4.1.** It suffices to show that the logarithm

$$\log F_{1a}(x, \xi) = \sum_{n=0}^{+\infty}\{\log f'(x_n) - \log f'(a_n)\}$$

is infinitely-differentiable.

The term by term derivative with respect to $x$ is the series

$$\sum_{n=0}^{+\infty} \frac{f''(x_n)x_n'}{f'(x_n)},$$

and it will be convenient to denote the $n$-th term by

$$T_n(x) = \frac{f''(x_n)x_n'}{f'(x_n)},$$

and the $n$-th partial sum by

$$S_n(x) = \sum_{j=0}^{n-1} T_j(x).$$

It will suffice to show that for each nonnegative integer $k$, $S_n^{(k)}(x)$ converges uniformly on $I_a$.

For any smooth function $\rho : (0, +\infty) \to (0, +\infty)$, and $k \in \mathbb{N}$, let us define $A_k(\rho)$ as the function

$$A_k(\rho) = \frac{d^k}{dx^k}\left(\frac{f''(\rho)\rho'}{f'(rho)}\right) - \frac{f''(\rho)\rho^{k+1}}{f'(\rho)}.$$

Then a straightforward induction establishes that $A_k(\rho)(x)$ is the sum of $M_k$ terms (where the integer $M_k$ depends on $k$, but not on $\rho$), each of which is a finite product

$$\frac{\gamma \prod_i f^{(r_j)}(\rho(x)) \prod_j \rho^{(t_j)}(x)}{(f'(\rho(x)))^k,}$$

where the coefficients $\gamma$ are fixed integers independent of $f$, where each $r_i \leq k+2$, each $t_j \leq k$, and at least one $t_j$ is present.

The term $T_n^{(k)}$ takes the form

$$A_k(x_n) + \frac{f''(x_n)x_n^{(k+1)}}{f'(x_n)}$$

.

To begin with, we observe that by the last two lemmas

$$\left| \frac{f''(x_n)x'_n}{f'(x_n)} \right| \leq KD_2D_{-1}|x_{n+1} - x_n|, \qquad \forall x \in I_a$$

hence $\{S_n(x)\}$ itself converges uniformly on $I_a$, with the error in $S_n(x)$ bounded by $KD_2D_{-1}a_n$, where $a_n = f^{\circ n}(a)$.

Now we will proceed by induction on $k$, and we first consider the first derivatives $T'_n(x)$ and note that

$$T'_n(x) = A_1(x_n) + B_1(x_n),$$

where

$$A_1(x_n) = \left\{ \frac{f'''(x_n)(x'_n)^2}{f'(x_n)} - \frac{(f''(x_n)x'_n)^2}{(f'(x_n))^2} \right\}$$

and

$$B_1(x_n) = \sum_{j=0}^{n-1} \frac{f''(x_n)x''_n}{f'(x_n)}.$$

Estimating each of its terms by its maximum, we see that $A_1(x_n)$ is dominated by

$$K^2(D_3D_{-1} + D_2^2D_{-1}^2)|x_{n+1} - x_n| \leq K|x_{n+1} - x_n|,$$

for a (different) constant $K$.

A calculation yields $x''_n = x'_nS_n$, so the term $B_1(x_n)$ is dominated by

$$K^2D_2D_{-1}|x_{n+1} - x_n|,$$

and we conclude that $S'_n(x)$ also converges uniformly on $I_a$, with error bounded by $Kx_n$.

We also observe that $|S'_n(x)| \leq Kx$.

Now we formulate an **induction hypothesis** $P_k$:

*There exist a constant $K$, depending only on $f$, $a$, and $k$, such that*

*(a) for $0 \leq j \leq k-1$, and each $n \geq 0$,*

$$|T_n^{(j)}(x)| \leq K|x_{n+1} - x_n|, \textit{ and } |S_n^{(j)}(x)| \leq K, \textit{ and}$$

*(b) for $1 \leq j \leq k$,*

$$|x_n^{(j)}| \leq K|x_{n+1} - x_n|.$$

We have established $P_2$.

Suppose $P_k$ holds, for some $k \geq 2$. Differentiating the formula $x_n'' = x_n' S_n$ $k-1$ times we get

$$x_n^{(k+1)} = \sum_{j=0}^{k-1} \binom{k-1}{j} x_n^{(j+1)} S_n^{(k-j-1)}$$

so conditions (a) and (b) of the hypothesis yield

$$|x_n^{(k+1)}| \leq \sum_{j=0}^{k-1} \binom{k-1}{j} K|x_{n+1} - x_n| = K|x_{n+1} - x_n|$$

(with a new $K$), and condition (b) of $P_{k+1}$ is proven.

Condition (a) then follows because of the form of $T_n^{(k)}$.

Thus, by induction, $P_k$ holds for each $k \geq 2$.

Thus $S_n^{(k)} = \sum T_m^{(k)}$ converges uniformly for all $k$, and Lemma 4.1 is proved. ∎

We now consider the three-parameter initial-value problem: $D_1(a, \alpha, \lambda)$ (cf. equation (2)).

It follows from Lemma 4.1 and standard results about ordinary differential equations [BDP, p.22] that problem $D_1(a, \alpha, \lambda)$ has a unique infinitely-differentiable solution $\phi(a, \alpha, \lambda; x)$ near $x = a$ whenever (P) holds, $a, \alpha > 0$, and $\lambda > 0$. Obviously, the solution is a strictly-increasing function of $x$ and its domain is an open subinterval of $(0, +\infty)$, containing $a$.

Note that

$$\prod_{j=n}^{m} f'(x_j) = (f^{\circ(m-n)})'(x_n) \approx \frac{x_{m+1} - x_m}{x_{n+1} - x_n},$$

so the product tends to 0 as $m \to +\infty$. It follows that the product $H_1(x, y)$ does not extend continuously to the closed quadrant $[0, +\infty) \times [0, +\infty)$, nor even to the corner $(0, 0)$, so there is no point in considering the differential equations at the endpoint. In fact, a moment's thought reveals that $H_1(x, y)$ tends to $\infty$ as $x \to 0$ for fixed $y > 0$, and tends to 0 as $y \to 0$ for fixed $x > 0$, so all positive numbers may be obtained as limits of $H_1(x, y)$ for suitable approach to $(0, 0)$ from inside $J \times J$.

The following lemma reformulates the information in the proof of Lemma 3.1 in the new notation:

**Lemma 4.5** *Suppose* $f, g, h \in \mathrm{D}([0, +\infty))$ *and* $f = g^h$. *Then* $\phi = h$ *is the solution to problem* $D_1(a, h(a), h'(0))$, *whenever* $a > 0$. ∎

To characterise the existence of a conjugating $h$, we need to formulate the conditions of this lemma in a way that does not refer explicitly to $h$.

Not all solutions to the initial-value problems $D_1(a, \alpha, \lambda)$ will be conjugating maps. For a start, we would need to ensure the condition $\phi(f(a)) = g(\alpha)$.

This leads us to consider the following:

**Lemma 4.6** *Assume condition (P), with $f, g \in S_-$. Then for each $a > 0$, and each $\alpha > 0$, there exists $\lambda > 0$ such that the solution to problem $D_1(a, \alpha, \lambda)$ has $\phi(f(a)) = g(\alpha)$.*

**Proof.** Given $a$ and $\alpha$, we could start by trying $\lambda = 1$. If the solution $\phi_1$ to $D_1(a, \alpha, 1)$ has $\phi_1(f(a)) = g(\alpha)$, we take $\lambda = 1$ and are done. If $\phi_1(f(a)) > g(\alpha)$, then decreasing $\lambda$ eventually reduces $\phi'$ to very small values on the interval $[a, f(a)]$, and hence pulls $\phi(f(a))$ above $g(\alpha)$. Thus, since $\phi(f(a))$ varies continuously with $\lambda$, there exists some $\lambda$ with $\phi(a, \alpha, \lambda)(f(a)) = g(\alpha)$, and we are done. If $\phi(f(a)) < g(\alpha)$, then we can attain a similar result by increasing $\lambda$ instead, because this increases $\phi'$ to very large values. ∎

Now we proceed to show that the solution $\phi$ of Lemma (4.6) is actually defined on the whole interval $(0, +\infty)$ and conjugates $f$ to $g$. We need another lemma first:

**Lemma 4.7** *Suppose that $u$ is a differentiable real-valued function on an open interval $U$, and for some constant $\kappa > 0$ we have*

$$|u'(x)| \leq \kappa \cdot |u(x)|, \quad \forall x \in U.$$

*Suppose that $u$ has a zero in $U$. Then $u$ is identically zero on $U$.*

**Proof.** The set $Z = u^{\circ -1}(0)$ of zeros of $u$ in $U$ is relatively-closed, and nonempty, so it suffices to show that it is open. Fix $a \in Z$, and choose $\epsilon > 0$ so that $a \pm \epsilon \in U$ and $\epsilon\kappa < 1$. Let $M$ be the maximum of $|u|$ on the closed interval $J = [a - \epsilon, a + \epsilon]$.

If $M > 0$, then choose $b \in J$ with $|u(b)| = M$. By the Law of the Mean, we may choose $c$ between $a$ and $b$ with $|u(b)| = |u'(c)| \cdot |b - a|$. But then

$$M = |u(b)| \leq \kappa M \cdot \epsilon < M,$$

which is impossible.

Thus $M = 0$, so $a$ is an interior point of $Z$.

Thus $Z$ is open, and we are done. ∎

**Lemma 4.8** *Suppose (P). Fix $a, \alpha > 0$. Choose $\lambda > 0$ such that the solution $h(x) = \phi(a, \alpha, \lambda; x)$ to problem $D_1(a, \alpha, \lambda)$ has $h(f(a)) = g(\alpha)$. Then the domain of the solution is $J = (0, +\infty)$, $h$ maps $J$ onto $J$ and $g \circ h = h \circ f$ on $J$.*

**Proof.** We establish that on each compact subinterval of $J$ we have an inequality $|u'| \leq \kappa \cdot |u|$, where

$$u(x) = g(\phi(x)) - \phi(f(x)).$$

In detail, one calculates (by fiddling with products) that

$$u'(x) = \lambda\{H_1(x,\phi(x)) \cdot g'(\phi(x)) - H_1(x, g^{-1}(\phi(f(x)))) \cdot g'(g^{-1}(\phi(f(x))))\},$$

and (using the Law of the Mean) estimates this (on a compact subinterval of $J$) by

$$\begin{aligned} &\kappa_1 \times |g^{-1}(\phi(f(x))) - \phi(x)| \\ \le\ &\kappa_2|\phi(f(x)) - g(\phi(x))| = \kappa_2|u(x)|. \end{aligned}$$

Then we apply Lemma 4.7 and the fact that $u(a) = g(\alpha) - \phi(f(a)) = 0$. This tells us that $u(x) = 0$ on the domain of $\phi$, which is the largest open interval $U$ with $a \in U$ on which $\phi(x) \in J$. But if either end (say $c$) of $U$ lies inside $J$, then by continuity $\phi$ is the solution to $D_1(c, \phi(c), \lambda)$, and $g(\phi(c)) = \phi(f(c)) \in J$, so $\phi$ extends to a neighbourhood of $c$, a contradiction. Thus $U = J$, and $\phi$ conjugates $f$ to $g$ on the whole of $J$. ∎

These results tell us that the initial-value problem together with the conjugation equation at one point are enough to guarantee the conjugation equation on the whole interval $J = (0, +\infty)$.

**Lemma 4.9** *Suppose condition (P) holds. If* $\phi : [0, +\infty) \to [0, +\infty)$ *satisfies*

$$\left.\begin{array}{rcl} \phi(f(x)) &=& g(\phi(x)), \\ \phi'(x) &=& H_1(x, \phi(x))\lambda \end{array}\right\} \qquad \forall x \in J,$$

*then* $\lim_{x\to d} \phi'(x) = \lambda$ *and* $\phi$ *has a one-sided derivative at* $d$*, equal to* $\lambda$.

**Proof.** Fix some $a \in J$ and denote $I_a = [f(a), a]$.
For fixed $x \in I_a$, letting $x_n = f^{\circ n}(x)$, we have

$$\begin{array}{rcl} \phi(f^{\circ n}(x)) &=& g^{\circ n}(\phi(x)), \\ \phi'(x_n) \cdot x_n' &=& g^{\circ n\prime}(\phi(x)) \cdot \phi'(x), \\ \phi'(x_n) &=& \prod_{j=0}^{n-1} \left( \dfrac{g'(g^{\circ j}(\phi(x))}{f'(x_j)} \right) \cdot \phi'(x), \end{array}$$

and this converges, uniformly on $I_a$, to $\lambda$. Since the product converges to $H_1(x, \phi(x))^{-1}$, the right-hand side converges to $\lambda$, so the derivative $\phi'$ extends continuously from $J$ to 0 if $\phi$ is given the value 0 there. This is enough to force the rest of the conclusions. ∎

Finally, we show that the $\lambda$ is unique:

**Lemma 4.10** *Suppose conditions (P) holds. Then, for each given* $a, \alpha \in J$*, there is* **exactly one** $\lambda > 0$ *for which the solution* $\phi = h$ *to problem* $D_1(a, \alpha, \lambda)$ *has* $h(f(a)) = g(\alpha)$*.*

**Proof.** Suppose this fails, and there are $\lambda_1 < \lambda_2$ such that the solutions $\phi_i$ to problems $D_1(a, \alpha, \lambda_i)$ $(i = 1, 2)$ both have $\phi_i(f(a)) = g(\phi_i(a))$.

Then by Lemma 4.8 both solutions have $\phi_i(f(x)) = g(\phi_i(x))$ on $J$, both map $J$ onto $J$, and both derivatives extend continuously to 0.

Since, initially, $\phi_1(a) = \phi_2(a)$ and $\phi_1'(a) < \phi_2'(a)$, we have $\phi_1(x) > \phi_2(x)$ for some distance to the left of $a$. Since $\phi_1(0) = \phi_2(0)(= 0)$, there exists a first point $e < a$ at which $\phi_1(e) = \phi_2(e)$. Just to the right of $e$, we have $\phi_1(x) > \phi_2(x)$, and hence $\phi_1'(e) \geq \phi_2'(e)$. But this contradicts the differential equation, because (since $\phi_1(e) = \phi_2(e)$) we have

$$\phi_1'(e) = \lambda_1 H(e, \phi_1(e)) < \lambda_2 H(e, \phi_2(e)) = \phi_2'(e).$$

This contradiction establishes the result. ∎

**Corollary 4.11** *Suppose Conditions (P) holds. Then there is precisely a one-parameter family of $C^1$ conjugations from $f$ to $g$ on $[0, +\infty)$.*

**Proof.** In fact, if we fix $a$, there is precisely one conjugation $\phi = \Phi_+(a, \alpha)$ for each $\alpha \in (0, +\infty)$. ∎

Thus there is *at most* a one-parameter family of $C^\infty$ conjugations from $f$ to $g$.

At this stage, we have completed the proof of the Main Theorem 2.2.

## 4.2 Remarks about $\phi'(0)$

Assume Conditions (P) holds. If 0 is a hyperbolic point for $f$, then the family of conjugating maps is parametrised by the multiplier at 0. If $f'(0) = 1$, but $f - x$ is not flat at 0, then it follows from Lemma 3.8 that all the conjugating maps have the same derivative at 0. This is seen by noting that the lemma, applied to the case $f = g$, shows that the $C^1$ centraliser of $f$ consists of maps that have derivative 1 at 0, and for general $g$ the family of conjugating maps from $f$ to $g$ is a coset of this centraliser.

Lemma 3.8 does not tell us anything about what happens when $f - x$ is flat at 0, but it is possible to see that again the conjugating $C^1$ maps all have the same derivative at 0. The essential point is the following, which can be proved more simply now than Lemma 3.8:

**Proposition 4.12** *Suppose $f \in S_-$, $f'(0) = 1$, and $\phi$ is a $C^1$ diffeomorphism of $[0, +\infty)$, commuting with $f$. Then $\phi'(0) = 1$.*

**Proof.** Fix $a > 0$, and let $\alpha = \phi(a)$. Then $\phi$ is $\Phi_+(a, \alpha)$. Let $a_k = f^{\circ k}(a)$ whenever $k \in \mathbb{Z}$. There is a unique $k$ such that

$$a_{k+1} \leq \alpha < a_k.$$

So at $a$, $\phi$ lies between $f^{\circ k}$ and $f^{\circ(k+1)}$.

If $\phi(a) = f^{\circ k}(a)$, then by Lemma 4.10, $\phi$ coincides with $f^{\circ k}$ on $J$, and hence has derivative 1 at 0, and we are done.

Otherwise, 4.10 tells us that $\phi$ never has the same value as $f^{\circ k}$ or $f^{\circ(k+1)}$ at any point, so its graph lies sandwiched between their graphs.

Thus
$$f^{\circ k}(x) - x > \phi(x) - x > f^{\circ(k+1)}(x) - x$$
for all $x > 0$, and hence, dividing by $x$ and taking limits we get $\phi'(0) = 1$. ∎

If we assume that the conjugating map is $C^\infty$ to 0, then Corollary 3.3 provides a much easier way to a stronger conclusion:

**Proposition 4.13 (Kopell)** *If $\phi \in \mathrm{D}([0,+\infty))$ commutes with $f$, and $f$ is flat at 0, then so is $\phi$.*

**Proof.** From the Corollary, $\phi(x) - x$ tends to zero more rapidly than any power of $x$, and hence *given that $\phi(x) - x$ is smooth*, all its derivatives vanish at 0. ∎

## 4.3 Remark about Centralisers

The special case $f = g$ of the foregoing corresponds to results of Kopell [K, pp. 167-71] about centralisers. Indeed, Kopell made use of the $f = g$ version of the differential equation of problem $D_1$ in order to obtain her results. See also [KCG, Section 8.6, pp. 353-5]. (We have not seen the differential equation for general $f$ and $g$ used in the literature.)

The elements of the centraliser $C_f$ of $f$ in $\mathrm{D}([0,+\infty))$ (where $f$ fixes only 0) are exactly the elements that conjugate $f$ to $f$, so applying the foregoing to the case $g = f$, we have Kopell's result that the centraliser is at most a one-parameter group. The centraliser is never trivial, since it has all iterates $f^{\circ n}$ ($n \in \mathbb{Z}$) as elements. However, it may fail to be connected. Sergeraert [SE, p.262-5] gave an example in which $f$ has no smooth compositional square root, and hence its centraliser is discrete.

Sergeraert also gave a useful sufficient condition for the centraliser of an element $f \in S_-$ to be connected. His condition is the existence of constants $\kappa > 0$ and $\delta > 0$ such that
$$\sup_{0 \le y \le x} (y - f(y)) \le \kappa(x - f(x)),$$
whenever $0 < x < \delta$. In particular, it always works if $x - f(x)$ is monotone.

The homomorphism $h \mapsto h'(0)$ maps the centraliser of a given $f$ to a multiplicative subgroup of $(0,+\infty)$, but (as we've seen) the subgroup in question is just $\{1\}$, as soon as $f'(0) = 1$.

In a rather similar way, the homomorphism $\Pi : h \mapsto T_0 h$ maps $C_f$ onto a subgroup of the group of invertible formal power series, and the image must have $T_0 f$ as an element.

We have seen in Proposition 4.13 that if $f - x$ is flat at 0, then all elements of its centraliser have the same property, so $\Pi$ is trivial.

Generally, the image of $C_f$ under $\Pi$ is a subgroup of the centraliser of $T_0 f$ in the power series group. In case $T_0 f = X \bmod X^{p+1}$ but $T_0 f \neq X \bmod X^{p+2}$, it is a purely algebraic fact (cf. [K, p.170], [L] or [KCG, p. 355ff]) that the latter centraliser is a one-parameter group, and indeed the map to the coefficient of $X^{p+1}$ is an isomorphism to $(\mathbb{R}, +)$.

It is interesting to note in passing that the differential equation provides a way to construct smooth compositional $k$-th roots of a diffeomorphism $f \in S^-$ of $[0, +\infty)$ that has a connected centraliser: One takes $f = g$, fixes $a > 0$, and considers the initial-value problem $D_1(a, \alpha, \Lambda_+(a, \alpha))$ for $\alpha$ between $a$ and $f(a)$. The solution $\phi_\alpha$ that has $\phi^{\circ k}(a) = f(a)$ is the desired root. Since $\phi^{\circ k}(a)$ moves continuously and monotonically away from $a$ as $\alpha$ moves towards $f(a)$ from $a$, and passes $f(a)$ before $\alpha$ reaches $f(a)$, there must exist a unique $\alpha$ with the above property.

## 4.4 Sufficiency of (P) and (T): Counterexample

The conditions (P) and (T) together are not sufficient for $C^\infty$ conjugacy, and the following example will demonstrate this.

We have noted that in the non-flat case the existence of a $C^1$-conjugacy is strictly weaker than the existence of a $C^\infty$ conjugacy. The example will also show that it is also weaker in the flat case.

Consider the diffeomorphisms of $[0, +\infty)$ defined on the interior by

$$\begin{aligned} f(x) &= x + e^{-1/x^2}, \\ \phi(x) &= x + x^{3/2}, \\ g &= f^\phi. \end{aligned}$$

One finds that $f$ and $g$ are smooth, but $\phi$ is only $C^1$: In fact, letting $\psi = \phi^{\circ -1}$, we calculate

$$\begin{gathered} \psi'(\phi)\phi' = 1, \\ \psi'(\phi)\phi'' + \psi''(\phi)(\phi')^2 = 0. \end{gathered} \tag{9}$$

Thus

$$\begin{gathered} g' = \psi'(f \circ \phi) f'(\phi)\phi', \\ g'' = \psi'(f \circ \phi) f'(\phi)\phi'' + \psi'(f \circ \phi) f''(\phi)(\phi')^2 + \psi''(f \circ \phi)\{f'(\phi)\phi'\}^2. \end{gathered}$$

The second term in the expression for $g''$ is continuous, and the other two add to

$$f'(\phi)\{\psi'(f \circ \phi)\phi'' + \psi''(f \circ \phi) f'(\phi)(\phi')^2\}. \tag{10}$$

The only problem is to see continuity at 0, and the point is that for small positive $x$ we have $\phi'(x) \approx 1$, $\psi'(x) \approx 1$,

$$\phi^{(k)}(x) = O(x^{\frac{3}{2}-k}), \forall k \geq 2$$

and for some sequence of integers $p_k$,

$$\psi^{(k)}(x) = O(x^{-p_k}), \forall k \geq 2$$

(as is verified inductively). Thus, since $f(x) - x$ is flat at 0, $f'(\phi(x))$ may be replaced by 1 and $f \circ \phi$ by $\phi$, in the expression (10), with an error that is $O(x^N)$ for all $N \in \mathbb{N}$. But when this is done we just get 0, by (9), so $g'' \to 0$ as $x \to 0$.

It now becomes clear that when we continue to differentiate $g$, and express $g^{(k)}$ in terms of $\psi, f$, and $\phi$, we get an expression involving derivatives of $\psi$ (at $f \circ \phi$), $f$ (at $\phi$), and $\phi$, and that when $f$ is replaced by $\mathbb{1}$ in this expression we get zero (the $k$-th derivative of $\psi \circ \phi$). Moreover, for small $x$, the error involved in replacing $f(\phi)$ by $\phi$, $f'(\phi)$ by 1, $f''(\phi)$ by 0, and all higher derivatives $f^{(k)}(\phi)$ by 0, is $O(x^N)$ for all $N$. Thus $g^{(k)} \to 0$ as $x \to 0$ for all $k \geq 3$, as well. It follows that $g$ is $C^\infty$, and $g(x) - x$ is flat at 0, as required.

Now any other $C^1$ conjugation of $f$ and $g$ will differ from $\phi$ by composition with an element of the centralizer of $f$. Since $f(x) - x$ is monotone, it satisfies Sergeraert's condition [S, p.259, Th.3.1], and hence the centralizer of $f$ consists of $C^\infty$ diffeomorphisms, and hence no conjugation of $f$ to $g$ is better than $C^1$.

This shows that Conditions (P) and (T) are not sufficient, by themselves, to guarantee conjugacy, in general.

**Question.** Since not all $C^1$ conjugacies between a given $f$ and $g$ belonging to $\mathrm{D}([0, \infty))$ are $C^\infty$ to zero, it would be interesting to know whether or not the set of parameters $\alpha$ for which the solution $\Phi_+(a, \alpha)$ is $C^\infty$ to zero is always a relatively closed subset of $(0, \infty)$. We were not able to resolve this question.

# 5 Compact Intervals

## 5.1 The Shape Condition

Now we consider conjugacy for orientation-preserving diffeomorphisms of a *compact* interval $I = [d, c]$, which are fixed-point-free on the interior $J$.

Applying the results about half-open intervals to both $[d, c)$ and $(c, d]$, we get:

**Lemma 5.1** *Suppose $I$ is compact, $f \in S_-$, and $f = g^h$ in $\mathrm{D}(I)$. Then $H_1(x, h(x))$ converges to $h'(x)/h'(d)$ for each $x \in J$, and $H_2(x, h(x))$ converges to $h'(x)/h'(c)$ for each $x \in J$.* ∎

Thus:

**Corollary 5.2** *If $f = g^h$, then the two-sided product*

$$H(x, h(x)) = \prod_{n=-\infty}^{\infty} \frac{f'(f^{\circ n}(x))}{g'(g^{\circ n}(h(x)))}$$

*is independent of $x \in J$, and equals the ratio $h'(c)/h'(d)$ of the derivatives of the conjugating map at the ends.* ∎

This immediately yields the first part of Proposition 2.4, that $H(x, h(x))$ is constant. The second part then follows from equation (5).

### 5.2 Remark

We close by noting the following characterisation of flowability, which is now clear.

**Proposition 5.3** *For compact intervals $I = [d, c]$, a diffeomorphism $f \in \mathrm{D}(I)$ that is fixed-point-free on $(c, d)$ embeds in a flow if and only if the centralisers of $f$ in $\mathrm{D}(\{c\} \cup J)$ and $\mathrm{D}(J \cup \{d\})$ are both connected, and coincide (when restricted to $J$).*

■

# References


[ALY] V. Afraimovitch, W.S. Liu and T. Young. Conventional multipliers for homoclinic orbits. Nonlinearity 9 (1996) 115-36.

[AR] P. Ahern and J.-P. Rosay. Entire functions, in the classification of germs tangent to the identity, in one or two variables. Transactions AMS 347 (1995) 543-72.

[B] G.R. Belitsky. Smooth classification of one-dimensional diffeomorphisms with hyperbolic fixed points. Siberian Math. J. 27 (1986) 801-4. (Original: Sib. Mat. Zh. 27 (1986) 21-4.)

[BDP] W.E. Boyce and R.C. DiPrima. Elementary Differential equations and Boundary Value Problems. 2nd ed. Wiley 1969.

[L] J. Lubin. Nonarchimedean dynamical systems. Compositio Mathematica 94 (1994) 321-46.

[MS] W. de Melo and S. van Strien. One-Dimensional Dynamics. Springer. 1993.

[K] N. Kopell. Commuting diffeomorphisms. pp. 165-84 in J. Palis + S. Smale (eds) Global Analysis. PSPM XIV. AMS. 1970.

[Ka] E. Kasner. Conformal classification of analytic arcs or elements: Poincaré's local problem of conformal geometry. Transactions AMS 16 (1915) 333-49.

[KH] A. Katok and B. Hasselblatt. Introduction to the Modern Theory of Dynamical Systems. Cambridge University Press. 1995.

[KCG] M. Kuczma, B. Choczewski and R. Ger. Iterative Functional Equations. Cambridge University Press. 1990.

[M] H. Mirkil. Differentiable functions, formal power series, and moments. Proceedings AMS 7 (1956) 650-2.

[OF2] A.G. O'Farrell. Composition of involutive power series, and reversible series. Comput. Methods Funct. Theory 8 (2008) 173-93.

[OR] A.G. O'Farrell and M. Roginskaya. Reducing conjugacy in the full diffeomorphism group of $\mathbb{R}$ to conjugacy in the subgroup of orientation-preserving

maps. (Russian) Zapiski Nauchnyh Seminarov POMI, to appear. (English translation in J. Mathematical Sciences, to appear.)
[RO] J.W. Robbin. Unfoldings of discrete dynamical systems. Ergodic Theory and Dynamical Systems 4 (1984) 421-86.
[SE] F. Sergeraert. Feuilletages et difféomorphismes infiniment tangent à l'identité. Inventiones Math. 39 (1977) 253-75.
[S] S. Sternberg. Local $C^n$ transformations of the real line. Duke Math. J. 24 (1957) 97-102.
[T] F. Takens. Normal forms for certain singularities of vectorfields. Ann. Inst. Fourier 23 (1973) 163-95.
[Y] T.R. Young. $C^k$ conjugacy of one-dimensional diffeomorphisms with periodic points. Proccedings AMS 125 (1997) 1987-95.